

HOMOTOPY INVARIANCE OF NON-STABLE K_1 -FUNCTORS

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ABSTRACT. Let G be a reductive algebraic group over a field k , such that every semisimple normal subgroup of G has isotropic rank ≥ 2 , i.e. contains $(\mathbf{G}_m)^2$. Let K_1^G be the non-stable K_1 -functor associated to G (also called the Whitehead group of G). We show that $K_1^G(k) = K_1^G(k[X_1, \dots, X_n])$ for any $n \geq 1$. This implies that K_1^G is \mathbb{A}^1 -homotopy invariant on the category of regular k -algebras, if k is perfect. If k is infinite perfect, one also deduces that $K_1^G(R) \rightarrow K_1^G(K)$ is injective for any regular local k -algebra R with the fraction field K .

1. INTRODUCTION

Let G be an isotropic reductive algebraic group over a field k . We study the properties of the functor K_1^G on the category of commutative unital k -algebras R , defined as

$$K_1^G(R) = G(R)/E(R),$$

where $E(-)$ is the elementary subgroup of $G(-)$, i.e. the subgroup generated by the points of unipotent radicals of parabolic subgroups of G (see [PSt1]; for a field k , the group $E(k)$ is the Tits' $G(k)^+$). The functor K_1^G is called the *non-stable* (or *unstable*) K_1 -functor associated to G , or the *Whitehead group* of G .

If $G = \mathrm{GL}_n$, $n \geq 3$, we have $K_1^G(R) = \mathrm{GL}_n(R)/E_n(R)$, the quotient involved in the Bass' definition of the algebraic K_1 -functor $K_1(R) = \lim_n \mathrm{GL}_n(R)/E_n(R)$ [B], hence the name “non-stable K_1 -functor”.

In general, the functors K_1^G happen to share many of the nice properties of the functor K_1 . The study of $K_1^{\mathrm{GL}_n}$, respectively, goes back to Bass' founding paper [B], and to A. Suslin [Su]. Later on, the functors K_1^G were also introduced and studied mostly for split semisimple groups G (Chevalley groups), and for groups G of classical type, such as the special orthogonal group of an isotropic quadratic form. The case where $R = k$ is a field was studied in a somewhat separate stream in relation to the Kneser–Tits problem. The principal bibliography can be found in [A, G, GMV1, HV, PSt1, W].

We mostly address the case where G is a reductive algebraic group of arbitrary type, such that all semisimple normal subgroups of G have isotropic rank ≥ 2 , i.e. contain $(\mathbf{G}_m)^2$. We show that the functor K_1^G has the following properties.

- If G is defined over a commutative ring R , the natural sequence of pointed sets

$$1 \longrightarrow K_1^G(R) \xrightarrow{g \mapsto (g, g)} K_1^G(R[X]) \times K_1^G(R[X^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_1^G(R[X, X^{-1}])$$

is exact.

This is Theorem 5.1 of our paper. It was first obtained by A. Suslin [Su, Theorem 5.1] for $G = \mathrm{GL}_n$ ($n \geq 3$), and, using similar methods, by A. Suslin and V. Kopeiko for O_{2n} ($n \geq 3$) [SuK, Theorem 6.8] and Sp_{2n} ($n \geq 2$) [K78, Theorem 3.9]. Later E. Abe [A, Theorem 2.16] generalized it to almost all Chevalley groups. More precisely, Abe proved the same statement for all simply connected split simple groups of rank ≥ 2 and any commutative ring R , excluding the groups of type C_l over R with $2 \notin R^\times$, and all groups of type B_l and G_2 . Our proof is only partially inspired by Abe's, and does not rely on his results. In particular, we remove the above restrictions on the type in the split case.

- If G is defined over a field k , then for any $n \geq 1$, one has the isomorphism

$$K_1^G(k) \xrightarrow{\cong} K_1^G(k[X_1, \dots, X_n]).$$

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This is proved in Theorem 5.3 by induction on n , relying on the previous property, much in the same way as the respective result of Abe for Chevalley groups [A, Theorem 3.5]. The only differences are that the induction base $n = 1$ is provided by a theorem of C. Soulé [Sou] and B. Margaux [M], and later we use the isomorphism $K_1^G(k) \xrightarrow{\cong} K_1^G(k(X))$ [G], which holds if G is a simply connected semisimple group. Note that in this latter case, moreover, one deduces that $K_1^G(k)$ is isomorphic to K_1^G of a ring of Laurent polynomials in several variables over k (Corollary 5.4).

For $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$ somewhat stronger results were previously obtained in the above-mentioned works of Suslin and Kopeiko and in [GMV1, K95b, K99, K96].

- If k is moreover a perfect field, for any regular k -algebra R one deduces that

$$K_1^G(R) \xrightarrow{\cong} K_1^G(R[X]).$$

This is proved in Theorem 5.5. We use Popescu's theorem to reduce to the case of a regular k -algebra essentially of finite type. In this generality, the result follows from the previous properties and Lindel's lemma on étale neighbourhoods [L]. This approach is due to T. Vorst [V], who considered the case of GL_n ; the same argument was used by Abe [A] for Chevalley groups under the same restrictions on type as mentioned above, and for Sp_{2n} by V. Kopeiko in [K95a, K96]. Note that in these three cases the ground field k was not supposed to be perfect, since for split groups the case of a regular algebra essentially of finite type over a non-perfect field reduces to the case of that over \mathbb{F}_p [V, Proof of Th. 3.3].

M. Wendt [W, Prop. 4.8] suggested a way to extend Abe's result to Chevalley groups of types B_l , C_l and G_2 , using stabilization results of E. Plotkin (actually, even in case of an excellent Dedekind ring k), but his proof is known to be incomplete [Ste].

Note that the isomorphism $K_1^G(R) \xrightarrow{\cong} K_1^G(R[X])$ implies that $K_1^G(R)$ also coincides with the 1st Karoubi–Villamayor K -group of R with respect to G , as defined by J.F. Jardine [J] following S.M. Gersten [Ge]; see [W] or Lemma 2.4.

- If k is an infinite perfect field, and R is a local regular k -algebra, the natural map

$$K_1^G(R) \rightarrow K_1^G(K),$$

where K is the fraction field of R , is injective.

This follows from the previous statements by means of a general theorem of J.-L. Colliot-Thélène and M. Ojanguren, proved in [CTO]; see Theorem 5.7.

It should be possible to extend the results of the present paper to isotropic simply connected simple groups G which are defined over a semilocal regular ring R containing a (perfect, infinite) field k , and not over k itself, by means of the techniques employed in [PaStV]. We plan to address this question in the future.

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2. THE FUNCTOR K_1^G

Let G be a reductive algebraic group over a commutative ring A .

2.1. Elementary subgroup and Suslin's local-global principle. We recall the main result of [PSt1].

Let P be a parabolic subgroup of G . Since the base $\mathrm{Spec} A$ is affine, the group P has a Levi subgroup L_P [SGA3, Exp. XXVI Cor. 2.3]. There is a unique parabolic subgroup P^- in G which is opposite to P with respect to L_P (that is $P^- \cap P = L_P$, see [SGA3, Exp. XXVI Th. 4.3.2]). We denote by U_P and U_{P^-} the unipotent radicals of P and P^- respectively.

We define the *elementary subgroup* $E_P(A)$ corresponding to P as the subgroup of $G(A)$ generated as an abstract group by $U_P(A)$ and $U_{P^-}(A)$. Note that if L'_P is another Levi subgroup of P , then L'_P and L_P are conjugate by some element $u \in U_P(A)$ [SGA3, Exp. XXVI Cor. 1.8], hence $E_P(A)$ does not depend on the choice of a Levi subgroup or, respectively, of an opposite subgroup P^- . We suppress the particular choice of L_P in this context, and sometimes even write U_P^- instead of U_{P^-} .

We say that a parabolic subgroup P in G is *strictly proper*, if it intersects properly every normal semisimple subgroup of G . Equivalently, P is strictly proper, if for every maximal ideal m in A the image of P_{A_m} in G_i under the projection map is a proper subgroup in G_i , where $G_{A_m}^{ad} = \prod_i G_i$ is the decomposition of the adjoint semisimple group $G_{A_m}^{ad}$ into a product of simple groups. It was proved in [PSt1], that if G satisfies the following strong isotropy condition

- G contains a strictly proper parabolic P over A , and, for any maximal ideal m in A ,
- (E) all irreducible components of the relative root system of G_{A_m} are of rank ≥ 2
(that is, every normal semisimple subgroup of G_{A_m} contains $(\mathbf{G}_m)^2$),

then $E(A) = E_P(A)$ is independent of the choice of a strictly proper parabolic subgroup P , and in particular, is normal in G . The key step in the proof is to show that, under the above assumption (E), G satisfies what we call Suslin's local-global principle (see [Su, Th. 3.1] for the case of GL_n):

Suslin's local-global principle. Let A be a commutative ring, G a reductive group scheme over A , $E(A)$ the elementary subgroup of $G(A)$. Let $g(X) \in G(A[X])$ be such that $g(0) \in E(A)$ and $F_m(g(X)) \in E(A_m[X])$ for all maximal ideals m of A . Then $g(X) \in E(A[X])$.

Note that Suslin based his proof of the above statement for GL_n on the ideas of Quillen from [Q] (e.g. [Q, Lemma 1]). For the case of split (=Chevalley) groups the same result was obtained by Abe in [A, Th. 1.15]. R. Basu has treated certain isotropic reductive groups of classical type under the assumption that they are locally split ([Ba]; see also [BBR]).

The most general known result for reductive groups is as follows:

Lemma 2.1. [PSt1, Lemma 17] *Let A be a commutative ring, G a reductive group over A , satisfying the condition (E). Then Suslin's local-global principle holds for G .*

2.2. The functor K_1^G and its \mathbb{A}^1 -invariance. Assume that G over A satisfies (E) as above. We consider the functor $K_1^G(R) = G(R)/E(R)$ on the category of commutative A -algebras R . The normality of the elementary subgroup implies that K_1^G is a group-valued functor.

Note that we have natural localization maps $F_m : K_1^G(A) \rightarrow K_1^G(A_m)$. Then the Suslin's local-global principle translates as follows:

$$\begin{aligned} x \in \ker(K_1^G(A[X]) \xrightarrow{X \mapsto 0} K_1^G(A)) \text{ is trivial, if and only if} \\ F_m(x) \in K_1^G(A_m[X]) \text{ is trivial for every maximal ideal } m \text{ of } A. \end{aligned}$$

Note that we also have a natural map $K_1^G(A) \rightarrow K_1^G(A[X])$, induced by the embedding $A \rightarrow A[X]$. We will say that K_1^G is **\mathbb{A}^1 -invariant at A** , if this map is an isomorphism, or, equivalently, if

$$G(A[X]) = G(A)E(A[X]).$$

In Theorem 5.5 we show that K_1^G is \mathbb{A}^1 -invariant at A , if G is an isotropic simply connected simple algebraic group over a perfect field k , A is a regular k -algebra, and the relative root system of G is of rank ≥ 2 .

Note that if a reductive group G over a commutative ring A is isotropic, i.e. contains a proper parabolic subgroup P (it is reasonable to assume that P is strictly proper), but G does not necessarily satisfy (E), one can still consider the quotient

$$K_1^{G,P}(A) = G(A)/E_P(A).$$

If A is a local ring, one knows that $E_P(A)$ is also independent of the choice of P and normal in $G(A)$; see [SGA3, Exp. XXVI § 5] and [PSt1, Lemma 12]. However, if A is not local, $E_P(A)$ is not in general normal in $G(A)$, for example, if $G = \mathrm{SL}_2$. The Suslin's local-global principle is not true in this case. Also, the classical example of Cohn [C] says that $\mathrm{SL}_2(k[X_1, X_2]) \neq E_2(k[X_1, X_2])$. Since $\mathrm{SL}_2(k[X_1]) = E_2(k[X_1])$, this implies that $K_1^{\mathrm{SL}_2, P_1}$ is not \mathbb{A}^1 -invariant at $k[X_1]$.

One may ask if the subgroup $\hat{E}(A)$ of $G(A)$ generated by all $E_P(A)$, P a parabolic subgroup of G , provides a better definition of K_1^G if (E) is not satisfied. Unfortunately, we know that if $G = \mathrm{SL}_2$, $A = k[X_1, X_2]$, k a finite field, then again $\mathrm{SL}_2(k[X_1, X_2]) \neq \hat{E}_2(k[X_1, X_2])$ [GMV2, Th. 1.4] (see also [KrMC] for a more general result).

These are the reasons why in the present paper we mostly restrict our attention to groups of isotropic rank ≥ 2 .

2.3. Margaux–Soulé theorem. Let G be an isotropic simply connected simple semisimple algebraic group over a field k . B. Margaux showed in [M], extending the earlier results of C. Soulé [Sou] on Chevalley groups, that

$$(1) \quad G(k[X]) = G(k) \cdot \langle U_P(k[X]), P \text{ a minimal parabolic } k\text{-subgroup of } G \rangle.$$

Two comments are in order. First, this result in the above papers was, actually, a corollary of a more general result about buildings associated to such groups G ; we will not need it in full generality. Second, by [SGA3, Exp. XXVI, Cor. 5.2, 5.7] all minimal parabolic subgroups of G over k are conjugate to any fixed minimal parabolic subgroup P by elements of $E_P(k)$, hence, we can rewrite the above statement as

$$(2) \quad G(k[X]) = G(k)E_P(k[X]) \text{ for any minimal parabolic } k\text{-subgroup } P \text{ of } G.$$

We can use the standard reductions to extend this result to isotropic reductive groups.

Lemma 2.2. *Let G be a reductive algebraic group over a field k . Let G^{sc} to be the simply connected semisimple group isogenous to the algebraic derived subgroup $[G, G]$ of G . Let $A = k[X_1, \dots, X_n]$ for some $n \geq 0$. If one has $G^{sc}(A[X]) = G^{sc}(A)E_P(A[X])$ for a minimal parabolic k -subgroup P of G^{sc} , then $G(A[X]) = G(A)E_Q(A[X])$ for any minimal parabolic k -subgroup Q of G .*

Proof. Let Q be a minimal parabolic k -subgroup of G . It is enough to show that any element g that is in $\ker(G(A[X]) \xrightarrow{X \mapsto 0} G(A))$, belongs to $E_Q(A[X])$.

First let G be a possibly non-simply connected semisimple group over k , satisfying the conditions of the Lemma. There is a short exact sequence of algebraic groups

$$1 \rightarrow C \xrightarrow{i} G^{sc} \xrightarrow{\pi} G \rightarrow 1,$$

where C is a group of multiplicative type over k , G^{sc} is a semisimple simply connected group over k , and C is central in G^{sc} . Write the respective “long” exact sequences over $A[X]$ and A with respect to fppf topology. Adding the maps induced by the homomorphism $\rho : A[X] \rightarrow A$, $X \mapsto 0$, we obtain a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & C(A[X]) & \xrightarrow{i} & G^{sc}(A[X]) & \xrightarrow{\pi} & G(A[X]) \xrightarrow{\delta} H_{fppf}^1(A[X], C) \\ \parallel & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho \\ 1 & \longrightarrow & C(A) & \xrightarrow{i} & G^{sc}(A) & \xrightarrow{\pi} & G(A) \longrightarrow H_{fppf}^1(A, C) \end{array}$$

Here the rightmost vertical arrow is an isomorphism, for example, by [CTS, Lemma 2.4]. Take any $g \in \ker(\rho : G(A[X]) \rightarrow G(A))$. Then $\delta(g) = 1$, hence there is $\tilde{g} \in G^{sc}(A[X])$ with $\pi(\tilde{g}) = g$. Clearly, $\rho(\tilde{g}) \in C(A)$, and hence

$$\tilde{g} \in C(A) \cdot \ker(\rho : G^{sc}(A[X]) \rightarrow G(A)) \subseteq C(A)E_P(A[X]).$$

Since $\pi(E_P(A[X])) = E_Q(A[X])$, this proves the claim of the Lemma for G .

Now let G be any reductive group over k satisfying the conditions of the Lemma. Then there is a short exact sequence

$$1 \rightarrow [G, G] \rightarrow G \rightarrow T \rightarrow 1,$$

for a k -torus T . Here the group $[G, G]$ is the algebraic derived subgroup of G ; it is a semisimple group that satisfies the isotropy conditions of the Lemma, if G does. Moreover, it contains the unipotent radicals of all parabolic subgroups of G , hence the subgroups $E_Q(A[X])$ are the same for $[G, G]$ and G . Since $T(A[X]) = T(k[X_1, \dots, X_n, X]) \cong T(k) \cong T(A)$ (e.g. by étale descent), the exact sequence

$$1 \rightarrow [G, G](A[X]) \rightarrow G(A[X]) \rightarrow T(A[X])$$

finishes the proof. \square

Theorem 2.3 (Margaux–Soulé). *Let G be a reductive algebraic group over a field k , such that every normal semisimple subgroup of the algebraic derived group $[G, G]$ is isotropic. Then*

$$G(k[X]) = G(k) \cdot \langle U_P(k[X]), P \text{ a minimal parabolic } k\text{-subgroup of } G \rangle.$$

Proof. By Lemma 2.2 and (1) it is enough to show that the claim holds if G is a simply connected semisimple group. By the results of [SGA3], any such G is isomorphic to a finite direct product of simply connected semisimple k -groups of the form $R_{k'/k}(H)$, where k' is a finite separable field extension of k , $R_{k'/k}$ denotes the Weil restriction functor, and H a simply connected simple semisimple group over k' . Clearly, any H involved in the decomposition of G has to be isotropic, and minimal parabolic subgroups of H are in one-to-one correspondence with Weil restrictions of minimal parabolic subgroups of $R_{k'/k}(H)$. Since $R_{k'/k}(H)(k[X]) = H(k'[X])$ etc., (1) extends to arbitrary semisimple simply connected groups. \square

If the isotropic rank of every normal semisimple subgroup of G over a field k is ≥ 2 (i.e. the rank of every irreducible component of the root system of G with respect to a maximal split torus is ≥ 2), Theorem 2.3 means that, in our notation,

$$(3) \quad K_1^G(k) \cong K_1^G(k[X]).$$

2.4. Relation to the Karoubi–Villamayor K -theory. For any reductive group G over a commutative ring A , let $KV_1^G(A)$ denote the 1st Karoubi–Villamayor K -group of the functor G , as defined by Jardine in [J, §3] following Gersten [Ge]. Note that Jardine denotes the Karoubi–Villamayor K -functor by K_1^G , while we reserve this notation for our K_1 -functor. The following result is a straightforward extension to isotropic reductive groups of [W, Lemma 2.4] for Chevalley groups. Note that even for Chevalley groups, the groups $K_1^G(A)$ are in general non-abelian ([vdK], see also [HV]).

Lemma 2.4. *Let G be an isotropic reductive group over a commutative ring A (with 1) satisfying (E). There is an exact sequence (a coequalizer)*

$$K_1^G(A[X]) \xrightarrow{g \mapsto g(1)g(0)^{-1}} K_1^G(A) \rightarrow KV_1^G(A) \rightarrow 1,$$

where the first map is a map of pointed sets, while the second one is a group homomorphism.

In particular, if K_1^G is \mathbb{A}^1 -invariant at A , then $K_1^G(A) \cong KV_1^G(A)$ as groups.

Proof. Let p denote both maps $A[X] \rightarrow A$ and $G(A[X]) \rightarrow G(A)$ induced by $X \mapsto 0$, and ε denote both maps $A[X] \rightarrow A$ and $G(A[X]) \rightarrow G(A)$ induced by $X \mapsto 1$. As in [J], set $EA = \ker(p : A[X] \rightarrow A)$, and let \tilde{G} be the extension of the functor G to the category of not necessarily unital commutative A -algebras, defined by $\tilde{G}(R) = \ker(pr_A : G(A \oplus R) \rightarrow G(A))$, here R is any commutative non-unital A -algebra, and $A \oplus R$ is the direct sum of additive groups with multiplication given by $(\alpha, a) \cdot (\beta, b) = (\alpha\beta, \alpha b + \beta a + ab)$.

Recall that $KV_1^G(A) = \operatorname{coker}(\varepsilon : \tilde{G}(EA) \rightarrow \tilde{G}(A))$. Thus, there is a canonical group homomorphism $G(A) \cong \tilde{G}(A) \rightarrow KV_1^G(A)$. We have $E(A) \subseteq \varepsilon(\tilde{G}(EA))$, where $\tilde{G}(EA)$ is identified with its image in $\tilde{G}(A)$. Indeed, $\tilde{G}(EA) = \ker(G(A \oplus EA) \rightarrow G(A))$; we have $A \oplus EA \cong A[X]$, hence $\tilde{G}(EA) = \ker(p : G(A[X]) \rightarrow G(A))$. By [PSt1, Lemma 8] for any $g \in E(A)$ there is $g(X) \in E(A[X]) \subseteq G(A[X])$ such that $g(0) = 1$ and $g(1) = g$. Hence $E(A) \subseteq \varepsilon(\ker(G(A[X]) \rightarrow G(A)))$. Summing up, there is a correctly defined map $K_1^G(A) = G(A)/E(A) \rightarrow KV_1^G(A)$. Clearly, it is surjective.

Now we show the exactness at the $K_1^G(A)$ term. By [J, Lemma 3.5] the inclusion $A \rightarrow A[X]$ induces an isomorphism between $KV_1^G(A)$ and $KV_1^G(A[X])$. Consider the image of $g(1)g(0)^{-1} \in K_1^G(A)$ in $K_1^G(A[X])$ under the inclusion map. One readily sees that $g(1)g(0)^{-1} = (g(Y)g(0)^{-1})|_{Y=1}$ is in $\varepsilon_Y(\ker(p_Y : G(A[X, Y]) \rightarrow G(A[X])))$, where ε_Y, p_Y are the same as ε, p with respect to the free variable Y . Therefore, the image of $g(1)g(0)^{-1}$ in $KV_1^G(A[X])$ is trivial, which implies that it is in $\ker(K_1^G(A) \rightarrow KV_1^G(A))$. Now let $g \in G(A)$ be such that the image of g under $G(A) \rightarrow K_1^G(A) \rightarrow KV_1^G(A)$ is trivial. Then $g \in \varepsilon(\ker(p : G(A[X]) \rightarrow A))$. This means that there is $g(X) \in G(A[X])$ such that $g(0) = 1$ and $g(1) = g$. Then $g = g(1)g(0)^{-1}$ belongs to the image of the map $K_1^G(A[X]) \rightarrow K_1^G(A)$ in our exact sequence. \square

3. NOTATION AND GENERAL LEMMAS OVER RINGS

3.1. Relative roots and relative root subschemes. Let R be a commutative ring. Let G be an isotropic reductive group scheme over R , P a strictly proper parabolic subgroup of G . Recall that we

set

$$E_P(R) = \langle U_P(R), U_{P^-}(R) \rangle,$$

where P^- is any parabolic subgroup of G opposite to $P = P^+$, and U_P and U_{P^-} are the unipotent radicals of P and P^- respectively. The main theorem of [PSt1] states that $E_P(R)$ does not depend on the choice of a strictly proper parabolic subgroup P , as soon as for any maximal ideal M in R all irreducible components of the relative root system of G_{R_M} are of rank ≥ 2 . Under this assumption, we call $E_P(R)$ the elementary subgroup of $G(R)$ and denote it simply by $E(R)$.

Now we define the relative roots and relative root subschemes of G with respect to $P = P^+$ (actually, they also depend on the choice of P^- , but we omit it from the notation). See [PSt1, LSt] for more details.

Let $L = P^+ \cap P^-$ be the common Levi subgroup of P^+ and P^- . It was shown in [PSt1] that we can represent $\text{Spec}(R)$ as a finite disjoint union

$$\text{Spec}(R) = \coprod_{i=1}^m \text{Spec}(R_i),$$

so that the following conditions hold for $i = 1, \dots, m$:

- the root system of $G_{\overline{k(s)}}$ is the same for all $s \in \text{Spec } R_i$;
- the type of the parabolic subgroup $P_{\overline{k(s)}}$ of $G_{\overline{k(s)}}$ is the same for all $s \in \text{Spec } R_i$;
- if S_i/R_i is a Galois extension of rings such that G_{S_i} is of inner type, then for any $s \in \text{Spec } R_i$ the Galois group $\text{Gal}(S_i/R_i)$ acts on the Dynkin diagram D_i of $G_{\overline{k(s)}}$ via the same subgroup of $\text{Aut}(D_i)$.

From here until the end of this section, assume that $R = R_i$ for some i . Denote by Φ the root system of G , by Π a set of simple roots of Φ , by D the corresponding Dynkin diagram. Then the $*$ -action on D is determined by a subgroup Γ of $\text{Aut } D$. Let J be the subset of Π such that $\Pi \setminus J$ is the type of $P_{\overline{k(s)}}$ (that is, the set of simple roots of the Levi subgroup $L_{\overline{k(s)}}$). Then J is Γ -invariant. Consider the projection

$$\pi = \pi_{J,\Gamma}: \mathbb{Z}\Phi \longrightarrow \mathbb{Z}\Phi / \langle \Pi \setminus J; \alpha - \sigma(\alpha), \alpha \in J, \sigma \in \Gamma \rangle.$$

The set $\Phi_P = \pi(\Phi) \setminus \{0\}$ is called the system of *relative roots* with respect to the parabolic subgroup P . The *rank* of Φ_P is the rank of $\pi(\mathbb{Z}\Phi)$ as a free abelian group.

If R is a local ring and P is a minimal parabolic subgroup of G , then Φ_P can be identified with the relative root system of G in the sense of [SGA3, Exp. XXVI §7] (or [BT1] in the field case), see also [PSt1, St].

To any relative root $A \in \Phi_P$ one associates a finitely generated projective R -module V_A and a closed embedding

$$X_A: W(V_A) \rightarrow G,$$

where $W(V_A)$ is the affine group scheme over R defined by V_A , which is called a *relative root subscheme* of G . These subschemes possess several nice properties similar to that of elementary root subgroups of a split group, see [PSt1, Th. 2]. Although they are just closed subschemes of G and not subgroups, we have the following multiplication formulas:

$$(4) \quad X_A(v)X_A(w) = X_A(v+w) \prod_{i \geq 1} X_{iA}(q_A^i(v, w)),$$

where each $q_A^i: W(V_A) \times_{\text{Spec } R} W(V_A) = W(V_A \oplus V_A) \rightarrow W(V_{iA})$ is a homogeneous map of degree i .

Secondly, the closed subschemes X_A are invariant under the conjugation action of the Levi subgroup L . Namely, for any $g \in L(R)$ we have

$$gX_A(v)g^{-1} = \prod_{i \geq 1} X_{iA}(\varphi_{g,A}^i(v)),$$

where each $\varphi_{g,A}^i: W(V_A) \rightarrow W(V_{iA})$ is homogeneous of degree i . Note that if g is contained in the central subtorus $\text{rad}(L)(R)$, then $\varphi_{g,A}^1$ is multiplication by a scalar, and all $\varphi_{g,A}^i$, $i > 1$, are trivial; this follows from the first part of [PSt1, Th. 2].

Thirdly, the relative root subschemes are subject to certain commutator relations which generalize the Chevalley commutator formula. Namely, assume that $A, B \in \Phi_P$ satisfy $mA \neq -kB$ for any $m, k \geq 1$. Then there exists a polynomial map

$$N_{ABij} : V_A \times V_B \rightarrow V_{iA+jB},$$

homogeneous of degree i in the first variable and of degree j in the second variable, such that for any R -algebra R' and for any $u \in V_A \otimes_R R', v \in V_B \otimes_R R'$ one has

$$(5) \quad [X_A(u), X_B(v)] = \prod_{i,j>0} X_{iA+jB}(N_{ABij}(u, v))$$

(see [PSt1, Lemma 9]).

In a strict analogy with the split case, for any R -algebra R' we have

$$E_P(R') = \langle X_A(V_A \otimes_R R'), A \in \Phi_P \rangle$$

(see [PSt1, Lemma 6]).

For any $\alpha \in \Phi_P$, we denote by $U_{(\alpha)}$ the closed subscheme $\prod_{k \geq 1} X_{k\alpha}$ of G so that we have $U_{(\alpha)}(R') = \langle X_{k\alpha}(V_{k\alpha} \otimes_R R'), k \geq 1 \rangle$ for any R'/R (here $X_{k\alpha}$ is assumed to be trivial if $k\alpha \notin \Phi_P$). The notation here coincides with that of [BT1] in case of isotropic reductive groups over a field.

Now let I be any ideal of the ring R . We denote

$$G(R, I) = \ker(G(R) \rightarrow G(R/I)), \quad E^*(A, I) = G(R, I) \cap E(R), \quad E(I) = \langle X_\alpha(IV_\alpha), \alpha \in \Phi_P \rangle, \\ E(R, I) = E(I)^{E(R)} = \text{the normal closure of } E(I) \text{ in } E(R).$$

For any $\alpha \in \Phi_P$, by [SGA3, Exp. XXVI Prop. 6.1] there exists a closed connected smooth subgroup G_α of G such that for any $s \in \text{Spec } R$, $(G_\alpha)_{\overline{k(s)}}$ is the standard reductive subgroup of $G_{\overline{k(s)}}$ corresponding to root subsystem $\pi^{-1}(\{\pm\alpha\} \cup \{0\}) \cap \Phi$. The group G_α is an isotropic reductive group “of isotropic rank 1”, having two opposite parabolic subgroups $L \cdot U_{(\alpha)}$ and $L \cdot U_{(-\alpha)}$.

We denote by $E_\alpha(R)$ the subgroup of $G(R)$ generated by $U_{(\alpha)}(R)$ and $U_{(-\alpha)}(R)$. Note that we don't know if $E_\alpha(R)$ is normal in $G_\alpha(R)$, and, generally speaking, it depends on the choice of the initial parabolic subgroup of G . For any $\alpha \in \Psi$, $u \in V_\alpha$, $a \in E_\alpha(R)$ we set

$$Z_\alpha(a, u) = aX_\alpha(u)a^{-1}.$$

3.2. Factorization lemma for the elementary subgroup. We fix a commutative ring A and an isotropic reductive group G over A . Let P be a strictly proper parabolic subgroup of G . We assume that A is small enough so that the relative root subschemes with respect to P are correctly defined over this base, as in subsection 3.1 above; Ψ denotes the system of relative roots of G with respect to P . Assume that $\text{rank } \Psi \geq 2$. Then $E(A) = E_P(A)$ is normal in $G(A)$.

First we prove some extensions of Lemmas 15–17 of [PSt1].

Lemma 3.1. *Fix $s \in A$, and let $F_s : G(A[Z]) \rightarrow G(A_s[Z])$ be the localization homomorphism. For any $g(Z) \in E(A_s[Z], ZA_s[Z])$ there exist such $h(Z) \in E(A[Z], ZA[Z])$ and $k \geq 0$ that $F_s(h(Z)) = g(s^k Z)$.*

Proof. Let $S \subseteq A$ be the set of all powers of h in A . One can prove exactly as in [PSt1, Lemma 15], that for any $g(Z) \in E(A_s[Z], ZA_s[Z])$ there exist such $f(Z) \in E(A[Z], ZA[Z])$ and $s \in S$ that $F_h(f(Z)) = g(sZ)$. Indeed, in that Lemma, the localization was taken with respect to the subset S of the base ring A which was a complement of a maximal ideal, and not a set of powers of one element; but the only use of the fact that A_S was a local ring was that G_{A_S} contained a parabolic subgroup whose relative root system was of rank ≥ 2 ; and such a parabolic subgroup in our current case is already defined over A . \square

Lemma 3.2. *Fix $s \in A$. For any $g(X) \in E(A_s[X])$ there exists $k \geq 0$ such that $g(aX)g(bX)^{-1} \in F_s(E(A[X]))$ for any $a, b \in A$ satisfying $a \equiv b \pmod{s^k}$.*

Proof. Consider $f(Z) = g(X(Y + Z))g(XY)^{-1} \in E(A_s[X, Y, Z])$. Then $f(0) = 1$, so $f(Z) \in E(A_s[X, Y, Z], ZA_s[X, Y, Z])$. By Lemma 3.1 there exist $h(Z) \in E(A[X, Y, Z], ZA[X, Y, Z])$ and $k \geq 0$ such that $F_s(h(Z)) = f(s^k Z)$. We have $f(s^k Z) = g(X(Y + s^k Z))g(XY)^{-1}$. If $a - b = s^k t$, $t \in A$, then setting $Y = b$, $Z = t$, we deduce the claim of the Lemma. \square

Suslin's local-global principle is closely related to the following factorization lemma (see [Su, Lemma 3.7] for GL_n , [A, Lemma 3.2] for split groups), which was originally inspired by another step in the proof of Quillen's local-global principle for projective modules [Q, Theorem 1].

Lemma 3.3. *Let A, G be as above. Let $f, g \in A$ be such that $fA + gA = A$. If $x \in E(A_{fg})$, then there exist $x_1 \in E(A_f)$, $x_2 \in E(A_g)$ such that $x = F_g(x_1)F_f(x_2)$.*

Proof. By [PSt1, Lemma 8] we can find such $x(X) \in E(A_{fg}[X])$ that $x(0) = 1$ and $x(1) = x$. Then it's enough to find $x_1(X) \in E(A_f[X])$, $x_2(X) \in E(A_g[X])$ such that $x(X) = x_1(X)x_2(X)$. Since $x(0) = 1$, by Lemma 3.2 there exists such $k \geq 0$ that for any $a, b \in A_{fg}$ such that $a \equiv b \pmod{f^k}$, we have $x(aX)x(bX)^{-1} \in F_f(E(A_g[X]))$; and for any $a, b \in A_{fg}$ such that $a \equiv b \pmod{g^k}$, we have $x(aX)x(bX)^{-1} \in F_g(E(A_f[X]))$. Since $fA + gA = A$, we have $f^kA + g^kA = A$ as well. Hence $1 = f^ks + g^kt$ for some $s, t \in A$. Then we have

$$x(X) = x((f^ks + g^kt)X)x(g^ktX)^{-1}x(g^ktX)x(0 \cdot X)^{-1}.$$

By the above, we have $x((f^ks + g^kt)X)x(g^ktX)^{-1} \in F_f(E(A_g[X]))$ and $x(g^ktX)x(0 \cdot X)^{-1} \in F_g(E(A_f[X]))$. \square

3.3. Nisnevich gluing for K_1^G . We prove here a gluing property of K_1^G , which looks like a segment of the Mayer-Vietoris sequence for a distinguished Nisnevich square. It is a straightforward extension of [V, Lemma 2.4] and [A, Lemma 3.7] for split groups; we only replace the usual split root subgroups by relative root subschemes.

Lemma 3.4. *Let G be an isotropic reductive group over a commutative ring B , with a strictly proper parabolic subgroup P , such that the relative root system Φ_P (e.g. in the sense of [SGA3, Exp. XXVI, §7], if B is connected semilocal) has rank ≥ 2 everywhere on $\mathrm{Spec} B$.*

Assume that B be a subring of a commutative ring A , and $h \in B$ is a non-nilpotent element. Denote by $F_h : G(A) \rightarrow G(A_h)$ the natural homomorphism.

(i) If $Ah + B = A$ (i.e. the natural map $B \rightarrow A/Ah$ is surjective), then for any $x \in E(A_h)$ there exist $y \in E(A)$ and $z \in E(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$ (i.e. $B/Bh \rightarrow A/Ah$ is an isomorphism), and h is not a zero divisor in A , then the sequence of pointed sets

$$K_1^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_1^G(B_h) \times K_1^G(A) \xrightarrow{(g_1, g_2) \mapsto g_1 F_h(g_2)^{-1}} K_1^G(A_h)$$

is exact.

Proof. (i) Write $x = \prod_{i=1}^m X_{\beta_i}(c_i)$, $c_i \in A_h \otimes_k V_{\beta_i}$, $\beta_i \in \Phi_P$. We show that $x \in E(A)E(B_h)$ by induction on the number of non-trivial factors in x . If $x = 1$, there is nothing to prove. Otherwise set $x_1 = \prod_{i=1}^{m-1} X_{\beta_i}(c_i)$, so that $x = x_1 X_{\beta_m}(c_m)$. Denote $\beta_m = \beta$, $c_m = c$ for short. Write $x_1 = y_1 z_1$, $y_1 \in E(A)$, $z_1 \in E(B_h)$. Then we have $x = y_1 z_1 X_{\beta}(c)$, where $c \in V_{\beta} \otimes_k A_h$.

By Lemma 3.1, there exists $N \geq 0$ large enough, such that there is $y(Z) \in E(A[Z], ZA[Z])$ satisfying $F_h(y(Z)) = z_1 X_{\beta}(h^N Z) z_1^{-1}$. On the other hand, note that $Ah + B = A$ implies $Ah^n + B = A$ for any $n \geq 1$. Let $M \geq 0$ be such that $h^M c \in V_{\beta} \otimes_k A$. Then one can find $a \in V_{\beta} \otimes_k A$, $b \in V_{\beta} \otimes_k B$ such that

$$c = ah^N + h^{-M}b.$$

By (4) we have

$$\begin{aligned} X_{\beta}(c) &= X_{\beta}(ah^N)X_{\beta}(h^{-M}b), \quad \text{if } 2\beta \notin \Psi; \\ X_{\beta}(c) &= X_{\beta}(ah^N)X_{\beta}(h^{-M}b) \prod_{i \geq 2} X_{i\beta}(q_{\beta}^i(h^N a, h^{-M}b)) \quad \text{otherwise.} \end{aligned}$$

In the first case we conclude right away

$$x = y_1 z_1 X_{\beta}(c) = y_1 (z_1 X_{\beta}(ah^N) z_1^{-1}) z_1 X_{\beta}(h^{-M}b) \in E(A)E(B_h).$$

In the second case we repeat the procedure to obtain a suitable factorization of $X_{i\beta}(q_{\beta}^i(h^N a, h^{-M}b)) \in E(A_h)$, $i \geq 2$ (height induction).

(ii) Take $x_1 \in G(B_h)$, $x_2 \in G(A)$ such that $x_1 F_h(x_2)^{-1} \in E(A_h)$. Then by (i) we have $y_1 x_1 = F_h(y_2 x_2)$ for some $y_1 \in E(B_h)$, $y_2 \in E(A)$. By assumption, $Ah^n \cap B = Bh^n$ for any $n \geq 0$, and hence $F_h(A) \cap B_h = F_h(B)$ in A_h . Since F_h is injective, we have $y_2 x_2 \in G(B)$. The claim follows. \square

Corollary 3.5. *Let G be an isotropic reductive group over a commutative ring B , with a strictly proper parabolic subgroup P , such that the relative root system Φ_P (e.g. in the sense of [SGA3, Exp. XXVI, §7], if B is connected semilocal) has rank ≥ 2 everywhere on $\text{Spec } B$.*

Let $\phi : B \rightarrow A$ be a homomorphism of commutative rings, and $h \in B$, $f \in A$ non-nilpotent elements such that $\phi(h) \in fA^\times$ and $\phi : B/Bh \rightarrow A/Af$ is an isomorphism. Assume moreover that the commutative square

$$\begin{array}{ccc} \text{Spec } A_f & \xrightarrow{F_f} & \text{Spec } A \\ \downarrow \phi & & \downarrow \phi \\ \text{Spec } B_h & \xrightarrow{F_h} & \text{Spec } B \end{array}$$

is a distinguished Nisnevich square. Then the sequence of pointed sets

$$K_1^G(B) \xrightarrow{(F_h, \phi)} K_1^G(B_h) \times K_1^G(A) \xrightarrow{(g_1, g_2) \mapsto \phi(g_1) F_f(g_2)^{-1}} K_1^G(A_f)$$

is exact.

Proof. Since $A_f = A_{\phi(h)}$, we can assume that $f = \phi(h)$ from the start. We have the following commutative diagram.

$$\begin{array}{ccccc} B & \xrightarrow{\phi} & \phi(B) & \hookrightarrow & A \\ \downarrow F_h & & \downarrow F_f & & \downarrow F_f \\ B_h & \longrightarrow & \phi(B)_f = \phi(B)_f & \hookrightarrow & A_f \end{array}$$

Since $E(A_f) \subseteq F_f(E(A))E(\phi(B)_f)$ by Lemma 3.4 (i), and $\phi : E(B_h) \rightarrow E(\phi(B)_f)$ is surjective, we have

$$(6) \quad E(A_f) \subseteq F_f(E(A)) \cdot \phi(E(B_h)).$$

Take $x_1 \in G(B_h)$, $x_2 \in G(A)$ such that $\phi(x_1) F_f(x_2)^{-1} \in E(A_f)$. Then by (6) we have $\phi(y_1 x_1) = F_f(y_2 x_2)$ for some $y_1 \in E(B_h)$, $y_2 \in E(A)$. Since G is a sheaf in the Nisnevich topology, there is $z \in G(B)$ such that $\phi(z) = y_2 x_2$, $F_h(z) = y_1 x_1$. This implies the claim of the Lemma. \square

3.4. Elementary subgroup over a polynomial ring. The following lemma extends [A, Prop. 1.6, Prop. 1.8, Cor. 1.9].

Lemma 3.6. *Let A be a commutative ring, and let I be an ideal of A such that the projection $\pi : A \rightarrow A/I$ has a section $i : A/I \rightarrow A$, i.e. i is a homomorphism such that $\pi \circ i = \text{id}$. Set $B = i(A/I) \subseteq A$.*

Let G a reductive group scheme over A , and P a (proper) parabolic subgroup of G . Then $E_P^(A, I) = E_P(A, I) = E_P(I)^{E_P(B)}$, and $E_P(A) \cap G(B) = E_P(B)$. In particular, $E_P^*(A[X], XA[X]) = E_P(A[X], XA[X])$, and $E_P(A[X]) \cap G(A) = E_P(A)$.*

Proof. We can assume that the relative roots and root subschemes with respect to P are correctly defined over A , and hence over B . Let Φ_P be the relative root system of G with respect to P over A . Assume that $g = \prod_{i=1}^m X_{\beta_i}(u_i) \in E_P^*(A, I)$, for some $\beta_i \in \Phi_P$, $u_i \in V_{\beta_i}$, where V_{β_i} are the respective finitely generated projective A -modules. Since

$$\ker(\pi : V_{\beta_i} \rightarrow V_{\beta_i} \otimes_A A/I) = V_{\beta_i} \otimes_A I,$$

we have $u_i = t_i + v_i$, where $t_i = i(\pi(u_i)) \in V_{\beta_i} \otimes_A B$, $v_i = u_i - t_i \in V_{\beta_i} \otimes_A I \subseteq V_{\beta_i}$. By (4), we have

$$X_{\beta_i}(u_i) = X_{\beta_i}(t_i) X_{\beta_i}(v_i) \prod_{k>1} X_{k\beta_i}(w_{i,k}),$$

where each $w_{i,k} = q_{\beta_i}^k(t_i, v_i) \in V_{k\beta_i} \otimes_A I$, since $q_{\beta_i}^k$ is a homogeneous map. Therefore, $X_{\beta_i}(u_i) = X_{\beta_i}(t_i) h_i$, for some $h_i \in E_P(I)$.

Set $g_k = \prod_{i=1}^k X_{\beta_i}(t_i)$, $1 \leq k \leq m$. We have $g_m = 1$, since $\pi(g) = 1$. Then

$$g = \prod_{k=1}^m g_k h_k g_k^{-1} \in E_P(I)^{E_P(B)}.$$

The remaining claims of the lemma are clear. □

The following lemma extends [A, Lemma 3.6] and [V, Lemma 2.1].

Lemma 3.7. *Let A be a commutative ring, S a multiplicative subset of A . Let G be a reductive group scheme over A , and P a (proper) parabolic subgroup of G . If $G(A[X_1, \dots, X_n]) = G(A)E_P(A[X_1, \dots, X_n])$ for some $n \geq 1$, then $G(A_S[X_1, \dots, X_n]) = G(A_S)E_P(A_S[X_1, \dots, X_n])$ as well.*

Proof. Let $g(X_1, \dots, X_n) \in G(A_S[X_1, \dots, X_n])$. We can assume $g(0) = 1$. There exists $s \in S$ such that $g(sX_1, \dots, sX_n) \in G(A[X_1, \dots, X_n])$. Since $g(0) = 1$, we have $g(sX_1, \dots, sX_n) \in E_P(A[X_1, \dots, X_n])$, that is, we can write $g(sX_1, \dots, sX_n) = \prod X_{\beta_i}(u_i(X_1, \dots, X_n))$, for some $\beta_i \in \Phi_P$, $u_i(X_1, \dots, X_n) \in V_{\beta_i} \otimes_A A[X_1, \dots, X_n]$. Then

$$g(X_1, \dots, X_n) = g(s(s^{-1}X_1), \dots, s(s^{-1}X_n)) = \prod X_{\beta_i}(u_i(s^{-1}X_1, \dots, s^{-1}X_n)) \in E_P(A_S[X_1, \dots, X_n]).$$

□

3.5. Generators of the congruence subgroup $E(A, I)$. Let A be any commutative ring, G an isotropic reductive group over A , P a parabolic subgroup of G . We assume that the system of relative roots $\Psi = \Phi_P$ and the respective relative root subschemes are defined over A .

Let $\alpha \in \Psi$ be a relative root, we will denote by m_α the positive integer satisfying $\Psi \cap \mathbb{Z}\alpha = \{\pm\alpha, \pm 2\alpha, \dots, \pm m_\alpha\alpha\}$. For $a \in E_\alpha(A)$, $u_i \in V_{i\alpha}$, $1 \leq i \leq m_\alpha$, we define

$$Z_\alpha(a, u_1, \dots, u_{m_\alpha}) = a \left(\prod_{i=1}^{m_\alpha} X_{i\alpha}(u_i) \right) a^{-1}.$$

The following lemma extends [A, Prop. 1.4].

Lemma 3.8. *Let A, G be as above. For any ideal I of A , the group $E(A, I)$ is generated by $Z_\alpha(a, u_1, \dots, u_{m_\alpha})$ for all $\alpha \in \Psi$, $u_i \in IV_{i\alpha}$, $1 \leq i \leq m_\alpha$, and $a \in E_\alpha(A)$.*

Proof. Take any $\beta \in \Psi$, $c \in V_\beta$, and $\alpha \in \Psi$. It is enough to show that for any $a \in E_\alpha(A)$, $u_i \in IV_{i\alpha}$, $1 \leq i \leq m_\alpha$,

$$x = X_\beta(c)Z_\alpha(a, u_1, \dots, u_{m_\alpha})X_\beta(c)^{-1}$$

is a product of elements $Z_\gamma(c, v_1, \dots, v_{n_\gamma})$, where $\gamma \in \Psi$, and $v_i \in IV_{i\gamma}$ for all $1 \leq i \leq n_\gamma$. If β is collinear to α , then $x = Z_\gamma(c, v_1, \dots, v_{n_\gamma})$ for $\gamma = \frac{1}{\gcd(m_\alpha, m_\beta)}\alpha$. If β is non-collinear to α , then by Lemma 3.9 below we have $x \in Z_\alpha(a, u_1, \dots, u_{m_\alpha}) \cdot E(I)$. □

Lemma 3.9. *Let $\alpha, \beta \in \Psi$ be two non-collinear relative roots, I, J two ideals of A . Let $a \in E_\alpha(A)$, $t \in A'$, $u_i \in IV_{i\alpha}$, $1 \leq i \leq m_\alpha$, and $v \in tJV_\beta \subseteq JV_\beta \otimes_A A'$, for some commutative ring A'/A . Then*

$$X_\beta(v)Z_\alpha(a, u_1, \dots, u_{m_\alpha})X_\beta(v)^{-1} = Z_\alpha(a, u_1, \dots, u_{m_\alpha})y,$$

where y is a product of $X_\gamma(w)$, $\gamma = i\alpha + j\beta \in \Psi$, $i, j \in \mathbb{Z}$, $j > 0$ and $w \in t^j J^j IV_\gamma \subseteq V_\gamma \otimes_A A'$.

Proof. For any $k \in \mathbb{Z} \setminus \{0\}$ and $w \in V_{k\alpha}$ we have by the formula for inverse and Chevalley commutator formula

$$\begin{aligned} X_\beta(v)X_{k\alpha}(w) &= X_{k\alpha}(w)[X_{k\alpha}(w)^{-1}, X_\beta(v)]X_\beta(v) \\ &= X_{k\alpha}(w) \cdot \prod_{i,j>0} X_{ki\alpha+j\beta}(w_{ij}) \cdot X_\beta(v), \quad w_{ij} \in t^j J^j V_{ki\alpha+j\beta}. \end{aligned}$$

Moreover, if $w \in IV_{k\alpha}$, then all $w_{ij} \in t^j J^j I^i V_{ki\alpha+j\beta}$. Note that for any $k, k' \in \mathbb{Z} \setminus \{0\}$, $i \geq 0$ and $i' > 0$, $j > 0$ and $j' \geq 0$, the roots $ki\alpha + j\beta$ and $k'i'\alpha + j'\beta$ cannot differ by a negative integral factor,

and their positive linear combinations lie in the set $\mathbb{Z}\alpha + \mathbb{N}\beta$. Therefore, we can apply commutator formulas again to deduce

$$[a^{-1}, X_\beta(v)] = \prod_{i \in \mathbb{Z}, j > 0} X_{i\alpha+j\beta}(w_{ij}), \quad w_{ij} \in t^j J^j V_{i\alpha+j\beta}$$

(note that the root factors with the same root can be gathered together by extra commutations), as well as

$$[(\prod_{i=1}^{m_\alpha} X_{i\alpha}(u_i))^{-1}, X_\beta(v)] = \prod_{i \in \mathbb{Z}, j > 0} X_{i\alpha+j\beta}(s_{ij}), \quad s_{ij} \in t^j J^j IV_{i\alpha+j\beta}.$$

Then we have

$$\begin{aligned} X_\beta(v) Z_\alpha(a, u_1, \dots, u_{m_\alpha}) X_\beta(v)^{-1} &= X_\beta(v) a \cdot \prod_{i=1}^{m_\alpha} X_{i\alpha}(u_i) \cdot a^{-1} X_\beta(v)^{-1} \\ &= a[a^{-1}, X_\beta(v)] X_\beta(v) \cdot \prod_{i=1}^{m_\alpha} X_{i\alpha}(u_i) \cdot X_\beta(v)^{-1} [X_\beta(v), a^{-1}] a^{-1} \\ &= a[a^{-1}, X_\beta(v)] \cdot \prod_{i=1}^{m_\alpha} X_{i\alpha}(u_i) \cdot [(\prod_{i=1}^{m_\alpha} X_{i\alpha}(u_i))^{-1}, X_\beta(v)] \cdot [a^{-1}, X_\beta(v)]^{-1} a^{-1} \\ &= a \cdot \prod_{i=1}^{m_\alpha} X_{i\alpha}(u_i) \cdot [(\prod_{i=1}^{m_\alpha} X_{i\alpha}(u_i))^{-1}, \prod_{i \in \mathbb{Z}, j > 0} X_{i\alpha+j\beta}(w_{ij})] \cdot [\prod_{i \in \mathbb{Z}, j > 0} X_{i\alpha+j\beta}(w_{ij}), \prod_{i \in \mathbb{Z}, j > 0} X_{i\alpha+j\beta}(s_{ij})] \cdot \\ &\quad \cdot \prod_{i \in \mathbb{Z}, j > 0} X_{i\alpha+j\beta}(s_{ij}) \cdot a^{-1} \\ &= Z_\alpha(a, u_1, \dots, u_{m_\alpha}) a x a^{-1}, \end{aligned}$$

where $x = \prod_{i \in \mathbb{Z}, j > 0} X_{i\alpha+j\beta}(r_{ij})$, $r_{ij} \in t^j J^j IV_{i\alpha+j\beta}$. Applying Chevalley commutator formula again, one deduces the claim of the lemma. \square

The following lemma is an analogue of [A, Cor. 2.7].

Lemma 3.10. *Let A, G be as above. Let I be an ideal of A . Let $\alpha \in \Psi$ be a non-divisible relative root (i.e. all relative roots collinear to α are its integral multiples). Then any element $x \in E(A, I)$ can be presented as a product $x = x_1 x_2$, where $x_1 \in E_\alpha(A, I)$, and x_2 is a product of elements of the form $Z_\beta(a, u_1, \dots, u_{m_\beta})$, where β is non-collinear to α , $u_i \in IV_{i\beta}$, $1 \leq i \leq m_\beta$, and $a \in E_\beta(A)$.*

Proof. Follows by induction from Lemmas 3.8 and 3.9. \square

We will need one more technical lemma.

Lemma 3.11. *Let A be a local ring, I the maximal ideal of A . For any isotropic reductive group G over A with two opposite parabolic subgroups $P = P^+$ and P^- defined over A , having the common Levi subgroup $L_P = P^+ \cap P^-$ and unipotent radicals U_P^\pm , we have*

$$G(A, I) = U_P^+(I) \cdot L_P(A, I) \cdot U_P^-(I) = U_P^-(I) \cdot L_P(A, I) \cdot U_P^+(I).$$

In particular, $G(A, I) = E_P(A, I) L_P(A, I)$.

Proof. Let $\rho : A \rightarrow A/I$ be the quotient map. The product $\Omega = U_P^+ \times L_P \times U_P^-$ embeds into G as an open subscheme via the multiplication morphism (e.g. [SGA3, Exp. XXVI, 4.3.6]). If for $g \in G(A)$ one has $\rho(g) \in \Omega(A/I)$, then, since I is the maximal ideal of the local ring A , we have $g \in \Omega(A) = U_P^+(A) L_P(A) U_P^-(A)$. If, moreover, $\rho(g) = 1$, then

$$g \in (\ker(\rho) \cap U_P^+(A)) \cdot (\ker(\rho) \cap L_P(A)) \cdot (\ker(\rho) \cap U_P^-(A)) = U_P^+(I) L_P(A, I) U_P^-(I).$$

\square

4. A PRESENTATION OF THE ELEMENTARY SUBGROUP OVER LAURENT POLYNOMIALS

In this section we obtain the following decomposition of the elementary subgroup of an isotropic reductive group over the ring of Laurent polynomials over a local ring.

Theorem 4.1. *Let G be an isotropic reductive algebraic group over a local ring A , satisfying the condition (E). Then*

$$E(A[X, X^{-1}]) = E(A[X])E(A[X^{-1}])E(A[X]).$$

The crucial corollary, that we will use to deduce our main results in the next section, is the following

Corollary 4.2. *Under the assumptions of Theorem 4.1, let I be the maximal ideal of A . Then*

$$E^*(A[X, X^{-1}], I \cdot A[X, X^{-1}]) = E^*(A[X], I \cdot A[X])E^*(A[X^{-1}], I \cdot A[X^{-1}]).$$

4.1. Construction of certain automorphisms. First we prove the existence of two kinds of automorphisms of an isotropic reductive group over a connected semilocal ring. The proofs basically consist in putting together some observations from [SGA3, Exp. XXVI §7].

Lemma 4.3. *Let G be a reductive group scheme over a connected semilocal ring R , P a minimal proper parabolic subgroup of G , L a Levi subgroup of P , Φ_P the respective system of relative roots, $\alpha_i \in \Phi_P$ a simple relative root. Let $G^{ad} = G/\text{Cent}(G) \subseteq \text{Aut}(G)$ be the corresponding adjoint group.*

There is a closed embedding $\mathbf{G}_m \rightarrow G^{ad}$ over R , such that for any R -algebra R' , and any $t \in \mathbf{G}_m(R') = (R')^\times$, one has

- 1) $t(X_\alpha(u)) = X_\alpha(t^{m_i(\alpha)} \cdot u)$ for any $\alpha \in \Phi_P$, $u \in V_\alpha \otimes_R R'$;
- 2) $t|_L = \text{id}_L$.

Proof. Let P', L' be the parabolic subgroup and the Levi subgroup of G^{ad} which are the images of P, L under $G \rightarrow G^{ad}$. Denote Φ_P by Ψ . We use the results of [SGA3, Exp. XXVI §7]. Let S be the maximal split subtorus of L' , $\Lambda = \text{Hom}(S, \mathbf{G}_m)$, $\Lambda^* = \text{Hom}(\mathbf{G}_m, S)$. By [SGA3, Exp. XXVI Th. 7.4] there is a map $\Psi \rightarrow \Lambda^*$ which defines a root datum $(\Lambda, \Lambda^*, \Psi, \Psi^*)$. Since G^{ad} is a group of adjoint type, by the compatibility with the absolute root datum [SGA3, Exp. XXVI Th. 7.13] the lattice Λ is generated by Ψ . For any $t \in R'$, let $\chi_t \in \Lambda^*$ be such that $\chi_t(\alpha_i) = t$ and $\chi_t(\alpha) = 1$ for any other simple root α of Ψ . Then $\chi : \mathbf{G}_m \rightarrow S$, $t \mapsto \chi_t$, is the desired embedding. The action on the relative root subschemes X_α is the desired one by [PSt1, Th. 2]. \square

Lemma 4.4. *Let G be a reductive group scheme over a connected semilocal ring R , $P = P^+$ a minimal proper parabolic subgroup of G , L a Levi subgroup of P , P^- the corresponding opposite parabolic subgroup, U_P and U_{P^-} the unipotent radicals of P and P^- . There exists an element $n_P \in E_P(R)$ such that*

$$n_P L n_P^{-1} = L, \quad n_P U_P n_P^{-1} = U_{P^-}, \quad n_P U_{P^-} n_P^{-1} = U_P.$$

Proof. Let $S \subseteq L$ be a maximal split subtorus of G . By [SGA3, Exp. XXVI, 7.2] the subgroups P and P^- are conjugate by an element $n_P \in \text{Norm}_G(L)(R) = \text{Norm}_G(S)(L)$, hence $n_P L n_P^{-1} = L$ and $n_P U_P n_P^{-1} = U_{P^-}$. Since the characters of S on $\text{Lie}(P^-)$ are opposite to those on $\text{Lie}(P)$, we also have $n_P U_{P^-} n_P^{-1} = U_P$. Since one has the Galois decomposition $G(R) = U_P(R)U_{P^-}(R)U_P(R)L(R)$ by [SGA3, Th. 5.1], we can assume that $n_P \in E_P(R)$. \square

4.2. The setting. It is easy to see that to prove Theorem 4.1, it is enough to consider the case where G is an (absolutely) simple reductive group; thus we will restrict our attention to this case until the very end of this section. From now on, we fix the following notation. Let A be a **local** ring with the maximal ideal I and residue field $l = A/I$, and let $\rho : A \rightarrow l$ be the natural map. Let G a simple group scheme over A of isotropic rank at least 2.

Let S be a maximal split subtorus of G , $P = P^+$ a minimal parabolic subgroup of G , P^- an opposite subgroup, $L = \text{Cent}_G(S)$ their common Levi subgroup, U^\pm their unipotent radicals. Let Φ be the absolute root system of G , $\Psi = \Phi_P$ the root system with respect to P , S . We consider relative root subschemes $X_\alpha(V_\alpha)$, $\alpha \in \Psi$, defined as in [PSt1]. Let Ψ' be the set of non-multipliable roots in Ψ (i.e. such that $2\alpha \notin \Psi$).

Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a system of simple roots of Ψ . We write $\alpha = \sum_{i=1}^n m_i(\alpha) \alpha_i$, $m_i(\alpha) \in \mathbb{Z}$, for any $\alpha \in \Psi$. We denote by $\tilde{\alpha}$ the highest positive root of Ψ . We assume that the numbering of Π is chosen so that α_1 is a terminal vertex on the Dynkin diagram of Ψ , and $m_1(\tilde{\alpha}) = 1$; or, if such a vertex does not exist, $m_1(\tilde{\alpha}) = 2$ and α_1 is the unique root adjacent to $-\tilde{\alpha}$ in the extended Dynkin diagram of Ψ . Note that in the latter case $\tilde{\alpha}$ is the only positive root with $m_1(\tilde{\alpha}) = 2$; the respective standard maximal parabolic subgroup is called extraspecial. If Ψ has no multipliable roots, α_1 is a long root; if $\Psi = BC_n$, then α_1 is a root of middle length (hence, non-multipliable), and $\{\alpha_1, \dots, \alpha_{n-1}, 2\alpha_n\}$ is a system of positive roots for Ψ' .

We denote by P_1^\pm the opposite standard maximal parabolic subgroups of G corresponding to α_1 , by L_1 their common Levi subgroup, and by U_1^\pm their unipotent radicals.

Consider G as a group over the ring of Laurent polynomials $A[X, X^{-1}]$. By Lemma 4.3 there is an automorphism σ of $G(A[X, X^{-1}])$ such that

- $\sigma|_{L_1} = \text{id}$;
- $\sigma(X_\alpha(u)) = X_\alpha(X^{m_1(\alpha)}u)$ for any $\alpha \in \Psi$, $u \in V_\alpha$.

As in [A], we denote

$$\begin{aligned} M_+^* &= E^*(A[X], I \cdot A[X]), & M^* &= E^*(A[X^{-1}], I \cdot A[X^{-1}]), \\ M^* &= E^*(A[X, X^{-1}], I \cdot A[X, X^{-1}]). \end{aligned}$$

Recall that by Lemma 3.6 we have $E^*(A[X], XA[X]) = E(A[X], XA[X])$. By Lemmas 3.8 and 3.6, the group $E(A[X], XA[X])$ is generated by its subgroups $E_\alpha(A[X], XA[X])$, $\alpha \in \Psi$. The same results also hold for X^{-1} instead of X . From now on, we will use these facts without any further reference.

4.3. The automorphisms τ_α .

Lemma 4.5. *Let $\alpha \in \Psi$ be a relative root. The group scheme G over k contains a semisimple algebraic k -subgroup G'_α of isotropic rank 1, such that the subgroups $U_{(\pm\alpha)} = \prod_{i \geq 1} X_{i\alpha}$ are the unipotent radicals of the two opposite minimal parabolic subgroups $P^\pm \cap G'_\alpha$ of G'_α , and $L \cap G'_\alpha$ is their common Levi subgroup. The subgroup G'_α is either simple, or a Weil restriction of a simple isotropic group over a finite separable extension of k .*

If $\Psi = G_2, F_4, E_8$, and α is a long root, then there exists a root subsystem Θ_α of Ψ of type A_2 , containing α , and a split simply connected simple algebraic subgroup G_{Θ_α} of G over k of type A_2 , such that X_β , $\beta \in \Theta_\alpha$, are root subgroups of G_{Θ_α} .

Proof. We will construct the desired subgroups in the case where G is a simply connected simple group. If G is not simply connected, they are defined to be the images of the respective subgroups in the simply connected covering G^{sc} of G under the natural homomorphism $G^{sc} \rightarrow G$.

Recall that we have defined in 3.1 reductive k -subgroups G_α of G . Such subgroup is determined by the fact that its Lie algebra has the form

$$\text{Lie}(G_\alpha) = \text{Lie}(L) \oplus \bigoplus_{i \in \mathbb{Z}} \text{Lie}(G)_{i\alpha},$$

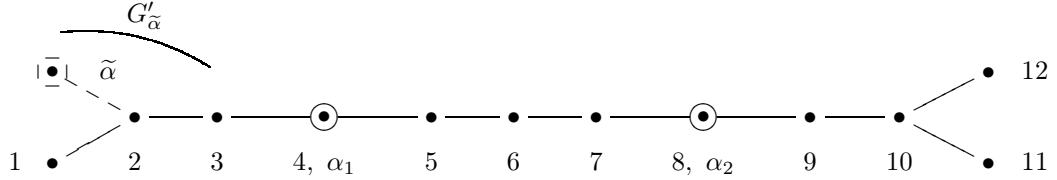
where $\text{Lie}(G)_{i\alpha}$ is the submodule of $\text{Lie}(G)$ corresponding to the character $i\alpha$ of S (see [SGA3, Prop. 6.1]). The derived subgroup of G_α is a semisimple group.

We can choose a basis Π of simple roots in Ψ so that α is either a simple root α_i , or equals $-\tilde{\alpha}$, if $\Psi = BC_n$. In the first case one readily sees that G_α is the Levi subgroup of the standard parabolic subgroup of G the (relative) type $\Pi \setminus \{\alpha\}$. It is well-known that the derived subgroup of a Levi subgroup of a parabolic subgroup of G is simply connected, if the ambient group G is. We take the semisimple indecomposable factor in the derived subgroup of G_α , that contains $U_{(\pm\alpha)}$, to be the group G'_α . If G is of outer type, the (absolute) Dynkin diagram of G'_α may consist of several connected components permuted by the $*$ -action. Then G'_α is a Weil restriction. Moreover, if $\Psi = G_2, F_4, E_8$, the classification of Tits indices [PSt2] shows that for any long root $\alpha \in \Psi$ the module V_α is 1-dimensional. Hence the corresponding relative root subgroups are the usual root subgroups of the split group G after a splitting base extension R'/R . One readily sees that for any pair of long roots $\alpha, \beta \in \Psi$ generating a subsystem of type $\Theta = A_2 \subseteq \Psi$, the respective factor in the derived subgroup of the reductive

subgroup of G defined by Θ via [SGA3, Prop. 6.1], is the derived subgroup of a Levi subgroup of a parabolic subgroup of $G_{R'}$, except in the case where G is a split group of type $\Psi = G_2$, where G_Θ is the long root subgroup (known to be simply connected). We choose the respective subgroups of type A_2 to be G_{Θ_α} .

If $\Psi = BC_n$, we can visualize the type of $G_\alpha = G_{\tilde{\alpha}}$ if we draw the extended absolute Dynkin diagram of G , and throw away all vertices corresponding to simple roots not belonging to L . In this case $G_{\tilde{\alpha}}$ may not be a Levi subgroup of any parabolic subgroup. However, one readily sees that in the groups of classical type, the connected component of the absolute Dynkin diagram D of $G_{\tilde{\alpha}}$, containing $-\tilde{\alpha}$, is the type of a simple factor of a Levi subgroup of a parabolic subgroup of G after a splitting base extension; hence it is the type of a simply connected simple group. See the list of possible Tits indices in [PSt2]. In the exceptional groups, this connected component always consists of one vertex, hence $X_{\pm\tilde{\alpha}}$ are 1-dimensional, and hence contained in a group of type SL_2 , a simple root subgroup after a splitting base extension. We choose the respective subgroup to be G'_α . \square

Example. Assume that G over k has absolute type ${}^1D_{12}$, the parabolic subgroup P is of type $\{4, 8\}$ (circled vertices on the Dynkin diagram below, standard Bourbaki numbering), $\Psi = BC_2$. Then $G_{\tilde{\alpha}}$ is a reductive group of type $D_4 + A_3 + D_4$. After a splitting base extension, the factor G''_β is contained in the Levi subgroup of the parabolic subgroup of G of type $\{4, 8\}$, but different from P ; it is standard with respect to the basis of simple roots obtained by adding $\tilde{\alpha}$ to the original one, and removing, say, the 12th root.



Let $\alpha \in \Psi$ be a relative root. Lemma 4.5 tells, in particular, that $P^\pm \cap G''_\alpha$ are the minimal parabolic subgroups of G''_α . Clearly, the respective relative root system of G''_α identifies with the subset $\{\pm\alpha, \pm 2\alpha, \dots, \pm m_\alpha\alpha\}$ of Ψ . By Lemma 4.3, there is an automorphism τ_α of $G''_\alpha(A[X, X^{-1}])$ such that $\tau_\alpha|_{L \cap G''_\alpha} = \text{id}$, and

$$\tau_\alpha(X_{k\alpha}(u)) = X_{k\alpha}(X^k u)$$

for any $k \geq 1$ and $u \in V_{k\alpha} \otimes_A A[X, X^{-1}]$.

Note that if $m_1(\alpha) = 1$, then $\sigma|_{E_\alpha(A[X, X^{-1}])} = \tau_\alpha$.

Lemma 4.6. *Let $\alpha \in \Psi'$ (i.e. α is non-multipliable). Then we have $\tau_\alpha^{\pm 1}(E_\alpha(A[X], XA[X])) \subseteq G'_\alpha(A[X]) \cap E(A[X])$.*

Proof. For the first statement we consider first τ_α , the case of τ_α^{-1} is symmetric. By Lemma 3.6, any $x \in E_\alpha(A[X], XA[X])$ is a product of $Z_{\pm\alpha}(a, Xf)$, where $a \in E_\alpha(A)$ and $f \in V_{\pm\alpha} \otimes_A A[X]$. By Lemma 4.4 there is $n_\alpha \in E_\alpha(A)$ such that $n_\alpha U_{(\alpha)} n_\alpha^{-1} \subseteq U_{(-\alpha)}$ and vice versa. Hence

$$Z_{-\alpha}(a, Xf) = a n_\alpha^{-1} (n_\alpha X_{-\alpha}(Xf) n_\alpha^{-1}) n_\alpha a^{-1} = a n_\alpha^{-1} X_\alpha(Xf') n_\alpha a^{-1} = Z_\alpha(a n_\alpha^{-1}, Xf'),$$

for some $f' \in V_\alpha \otimes_A A[X]$. Therefore, we only need to check that $\tau_\alpha(Z_\alpha(a, Xf)) \in E(A[X])$ for any $a \in E_\alpha(A)$, $f \in V_\alpha \otimes_A A[X]$.

By Gauss decomposition in $G'_\alpha(A)$ [SGA3, Théorème 5.1] we have

$$a = l X_\alpha(a_1) X_{-\alpha}(b) X_\alpha(a_2),$$

$a_1, a_2, b \in A$, $l \in L_\alpha(A)$. Then $\tau_\alpha(a) = l X_\alpha(a_1 X) X_{-\alpha}(b X^{-1}) X_\alpha(a_2 X)$. Clearly, it is enough to check that

$$X_{-\alpha}(b X^{-1}) X_\alpha(a_2 X) X_\alpha(X^2 f) (X_{-\alpha}(b X^{-1}) X_\alpha(a_2 X))^{-1} = X_{-\alpha}(b X^{-1}) X_\alpha(X^2 f) \in E(A[X]).$$

Note that α belongs to a root subsystem of Ψ of type A_2 , B_2 , or is a short root in G_2 . Assume first it belongs to a root subsystem of type A_2 . Then $X_\alpha(b X^2) = [X_\beta(uX), X_\gamma(vX)]$, $u \in V_\beta$, $v \in V_\gamma$,

$\beta + \gamma = \alpha$, β, γ non-collinear to α ([LSt, Lemma 2]). Then by the generalized Chevalley commutator formula both $X_{-\alpha}(bX^{-1})(X_\beta(uX)^{\pm 1})$ and $X_{-\alpha}(bX^{-1})(X_\gamma(vX)^{\pm 1})$ belong to $E(A[X])$. Therefore, $X_{-\alpha}(bX^{-1})X_\alpha(X^2f) \in E(A[X])$.

In the case of B_2 , if α is long, let β be a short root such that α, β is a system of simple roots for B_2 . By [LSt, Lemma 2 (2)] there are such $u \in V_{\alpha+\beta} \otimes_A A[X]$, $v \in V_{-\beta} \otimes_A A[X]$, and $c_i \in V_{\alpha+2\beta} \otimes_A A[X]$, $d_i \in V_{-\beta} \otimes_A A[X]$, $1 \leq i \leq k$, such that

$$f = N_{\alpha+\beta, -\beta, 1, 1}(u, v) + \sum_{i=1}^k N_{\alpha+2\beta, -\beta, 1, 2}(c_i, d_i).$$

By the generalized Chevalley commutator formula, this means that

$$X_\alpha(X^2f) = [X_{\alpha+\beta}(u), X_{-\beta}(v)] \prod_{i=1}^k ([X_{\alpha+2\beta}(c_i), X_{-\beta}(d_i)] X_{\alpha+\beta}(-X N_{\alpha+2\beta, -\beta, 1, 1}(c_i, d_i))).$$

Since a long root in B_2 cannot be added to another root more than once, and since $(-\alpha) + (\alpha + 2\beta)$ is not a root, we again have $X_{-\alpha}(bX^{-1})X_\alpha(X^2f) \in E(A[X])$ by the generalized Chevalley commutator formula.

If α is a short root in a subsystem of type B_2 , let β denote a long root in this B_2 such that α, β form a system of simple roots. By [LSt, Lemma 2 (1b)], since $(-\beta) - (\alpha + \beta)$ is not a root, we can write

$$X_\alpha(bX^2) = [X_{-\beta}(uX), X_{\alpha+\beta}(vX)] X_{2\alpha+\beta}(wX^3),$$

for some $u \in V_{-\beta}$, $v \in V_{\alpha+\beta}$, $w \in V_{2\alpha+\beta}$. By the generalized Chevalley commutator formulas, $X_{-\alpha}(bX^{-1})X_{2\alpha+\beta}(wX^3) \in E(A[X])$. On the other hand,

$$\begin{aligned} X_{-\alpha}(bX^{-1})[X_{-\beta}(uX), X_{\alpha+\beta}(vX)] &= [X_{-\alpha}(bX^{-1})X_{-\beta}(uX), X_{-\alpha}(bX^{-1})X_{\alpha+\beta}(vX)] \\ &= [X_{-\alpha-\beta}(c_1)X_{-2\alpha-\beta}(c_2X^{-1})X_{-\beta}(uX), X_\beta(c_3)X_{\alpha+\beta}(vX)], \end{aligned}$$

for some $c_1 \in V_{-\alpha-\beta}$, $c_2 \in V_{-2\alpha-\beta}$, $c_3 \in V_\beta$. Note that $X_{-2\alpha-\beta}(c_2X^{-1})$ commutes with all other root factors involved in the last expression, except for $X_{\alpha+\beta}(vX)$, and the commutator with the latter is equal

$$[X_{-2\alpha-\beta}(c_2X^{-1}), X_{\alpha+\beta}(vX)] = X_{-\alpha}(c_4)X_\beta(c_5X),$$

for some $c_4 \in V_{-\alpha}$, $c_5 \in V_\beta$. Thus, we can safely cancel the only negative factor $X_{-2\alpha-\beta}(c_2X^{-1})$ with its inverse. Therefore, $X_{-\alpha}(bX^{-1})[X_{-\beta}(uX), X_{\alpha+\beta}(vX)] \in E(A[X])$.

If α is a short root in a subsystem of type G_2 , let β denote a long root in this G_2 such that α, β form a system of simple roots. Since $(\alpha + \beta) - (-\beta)$ is not a root, by Lemma [LSt, Lemma 2 (1b)] and the generalized Chevalley commutator formula, we can write

$$X_\alpha(X^2f) = [X_{\alpha+\beta}(u), X_{-\beta}(X^2v)] X_{2\alpha+\beta}(X^2w_1) X_{3\alpha+2\beta}(X^2w_2) X_{3\alpha+\beta}(X^4w_3),$$

for some $u \in V_{\alpha+\beta} \otimes_A A[X]$, $v \in V_{-\beta} \otimes_A A[X]$, etc. One readily sees that by the generalized Chevalley commutator formula $X_{-\alpha}(X^{-1}b)X_{3\alpha+2\beta}(X^2w_2) = X_{3\alpha+2\beta}(X^2w_2)$, as well as $X_{-\alpha}(X^{-1}b)X_{2\alpha+\beta}(X^2w_1)$ and $X_{-\alpha}(X^{-1}b)X_{3\alpha+\beta}(X^4w_3)$, all belong to $E(A[X])$. On the other hand, we have

$$(7) \quad \begin{aligned} X_{-\alpha}(X^{-1}b)[X_{\alpha+\beta}(u), X_{-\beta}(X^2v)] &= [X_{\alpha+\beta}(u)X_\beta(X^{-1}c_1), X_{-\beta}(X^2v)X_{-\alpha-\beta}(Xc_2)X_{-2\alpha-\beta}(c_3) \\ &\quad \cdot X_{-3\alpha-\beta}(X^{-1}c_4)X_{-3\alpha-2\beta}(Xc_5)], \end{aligned}$$

where $c_1 \in V_\beta \otimes_A A[X]$ etc. Note that $X_{-3\alpha-\beta}(X^{-1}c_4)$ commutes with both $X_{\alpha+\beta}(-u)$ and $X_\beta(-X^{-1}c_1)$, since the sums of respective roots are not roots; hence we can cancel out the factor $X_{-3\alpha-\beta}(X^{-1}c_4)$ with its inverse in (7) without modifying anything else. Then we can also eliminate $X_\beta(X^{-1}c_1)$. Indeed, by the A_1 case considered above, we have $X_\beta(X^{-1}c_1)X_{-\beta}(X^2v) \in E(A[X])$; and by the generalized Chevalley commutator formula, $X_\beta(X^{-1}c_1)$ commutes with $X_{-2\alpha-\beta}(c_3)$, and its commutators with $X_{-\alpha-\beta}(Xc_2)$ and $X_{-3\alpha-2\beta}(Xc_5)$ belong to $E(A[X])$. This implies that the expression (7) belongs to $E([X])$, and hence we are done. \square

Observe that Lemma 4.6 does not cover the case where Ψ is of type BC_l and α is an extra-short root. To treat this case, we need first to prove the following preparatory lemmas.

Lemma 4.7. *Assume that $\Psi = BC_l$, $l \geq 2$. Let $\alpha, \beta \in \Psi$ be two simple roots in a root subsystem of type BC_2 , with α extra-short. Consider the subgroup*

$$Y_{\alpha, \beta}^+ = \langle X_{-\beta}(V_\beta \otimes_A XA[X]), X_{\pm(\alpha+\beta)}(V_{\pm(\alpha+\beta)} \otimes_A A[X]), \\ X_{\pm 2(\alpha+\beta)}(V_{\pm 2(\alpha+\beta)} \otimes_A A[X]), X_\beta(V_\beta \otimes_A A[X]), X_{-2\alpha-\beta}(V_{-2\alpha-\beta} \otimes_A A[X]), \\ X_{2\alpha+\beta}(V_{2\alpha+\beta} \otimes_A XA[X]) \rangle \subseteq E(A[X]).$$

Then

- (i) $Y_{\alpha, \beta}^+$ contains $X_{2\alpha}(V_{2\alpha} \otimes_A X^2A[X])$, $X_{-2\alpha}(V_{-2\alpha} \otimes_A A[X])$, $X_\alpha(V_\alpha \otimes_A XA[X])$, and $X_{-\alpha}(V_{-\alpha} \otimes_A A[X])$.
- (ii) $Y_{\alpha, \beta}^+$ is normalized by $X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X])$.

Proof. (i) By the generalized Chevalley commutator formula and [LSt, Lemma 2 (2)], for any $u \in V_{2\alpha} \otimes_A A[X]$, there are such $v \in V_{2\alpha+\beta} \otimes_A A[X]$, $w \in V_{-\beta} \otimes_A A[X]$, and $c_1, \dots, c_m \in V_{2(\alpha+\beta)} \otimes_A A[X]$, $d_1, \dots, d_m \in V_{-\beta} \otimes_A A[X]$, that

$$X_{2\alpha}(X^2u) = X_{2\alpha}(N_{2\alpha+\beta, -\beta, 1, 1}(v, w)) \prod_{i=1}^m X_{2\alpha}(N_{2(\alpha+\beta), -\beta, 1, 2}(c_i, Xd_i)) \\ = [X_{2\alpha+\beta}(Xv), X_{-\beta}(Xw)] \prod_{i=1}^m ([X_{2(\alpha+\beta)}(c_i), X_{-\beta}(Xd_i)] X_{2\alpha+\beta}(-XN_{2(\alpha+\beta), -\beta, 1, 1}(c_i, d_i))).$$

Hence $X_{2\alpha}(V_{2\alpha} \otimes_A X^2A[X]) \subseteq Y_{\alpha, \beta}^+$. On the other hand, by [LSt, Lemma 2 (1), case (b)], for any $u \in V_\alpha \otimes_A A[X]$ there are such $v \in V_{\alpha+\beta} \otimes_A A[X]$, $w \in V_{-\beta} \otimes_A A[X]$, that $u = N_{\alpha+\beta, -\beta, 1, 1}(v, w)$, i.e.

$$(8) \quad X_\alpha(Xu) = [X_{\alpha+\beta}(v), X_{-\beta}(Xw)] \cdot X_{2\alpha+\beta}(-XN_{\alpha+\beta, -\beta, 2, 1}(v, w)) \cdot X_{2\alpha}(-X^2N_{\alpha+\beta, -\beta, 1, 2}(v, w)).$$

Hence $X_\alpha(V_\alpha \otimes_A XA[X]) \subseteq Y_{\alpha, \beta}^+$.

The cases of $X_{-2\alpha}(V_{-2\alpha} \otimes_A A[X])$ and $X_{-\alpha}(V_{-\alpha} \otimes_A A[X])$ is treated in the same way, using the opposite roots.

(ii) Follows from the generalized Chevalley commutator formula. \square

Lemma 4.8. *Assume the hypothesis of Lemma 4.7. For any*

$$g \in X_{-\alpha}(V_{-\alpha} \otimes_A X^{-1}A[X])X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-2}A[X]),$$

the sets

- (i) $gX_{2\alpha+\beta}(V_{2\alpha+\beta} \otimes_A X^2A[X])g^{-1}$,
- (ii) $gX_\alpha(V_\alpha \otimes_A X^2A[X])g^{-1}$,
- (iii) $gX_{2\alpha}(V_{2\alpha} \otimes_A X^3A[X])g^{-1}$

are all contained in $Y_{\alpha, \beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X])$.

Proof. Set $g = g_2g_1$, $g_1 \in X_{-\alpha}(V_{-\alpha} \otimes_A X^{-1}A[X])$, $g_2 \in X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-2}A[X])$.

(i) Using the generalized Chevalley commutator formula, we obtain

$$g_2g_1X_{2\alpha+\beta}(V_{2\alpha+\beta} \otimes_A X^2A[X])g_1^{-1}g_2^{-1} \subseteq \\ \subseteq g_2 \left(X_\beta(V_\beta \otimes_A A[X])X_{\alpha+\beta}(V_{\alpha+\beta} \otimes_A XA[X]) \cdot \right. \\ \left. \cdot X_{2(\alpha+\beta)}(V_{2(\alpha+\beta)} \otimes_A X^2A[X])X_{2\alpha+\beta}(V_{2\alpha+\beta} \otimes_A X^2A[X]) \right) g_2^{-1} \\ \subseteq X_\beta(V_\beta \otimes_A A[X])X_{\alpha+\beta}(V_{\alpha+\beta} \otimes_A XA[X]) \cdot \\ \cdot X_{2(\alpha+\beta)}(V_{2(\alpha+\beta)} \otimes_A X^2A[X])X_{2\alpha+\beta}(V_{2\alpha+\beta} \otimes_A X^2A[X])X_\beta(V_\beta \otimes_A A[X]) \\ \subseteq Y_{\alpha, \beta}^+.$$

(ii) To compute $gX_{2\alpha}(V_{2\alpha} \otimes_A X^3A[X])g^{-1}$, we use [LSt, Lemma 2 (2)] to conclude that for any $u \in V_{2\alpha} \otimes_A A[X]$, there are $w \in V_{-\beta} \otimes_A A[X]$, $v \in V_{2\alpha+\beta} \otimes_A A[X]$, and $w_i \in V_{-\beta} \otimes_A A[X]$,

$v_i \in V_{2\alpha+2\beta} \otimes_A A[X]$, $1 \leq i \leq k$, such that $u = N_{-\beta, 2\alpha+\beta, 1, 1}(w, v) + \sum_{i=1}^k N_{-\beta, 2\alpha+2\beta, 2, 1}(w_i, v_i)$. Therefore, one has

$$(9) \quad X_{2\alpha}(X^3u) = [X_{-\beta}(Xw), X_{2\alpha+\beta}(X^2v)] \prod_{i=1}^k ([X_{-\beta}(Xw_i), X_{2\alpha+2\beta}(Xv_i)] X_{2\alpha+\beta}(X^2s_i)),$$

where $s_i = -N_{-\beta, 2\alpha+2\beta, 1, 1}(w_i, v_i)$, $1 \leq i \leq k$. First we treat the first commutator in (9). We have

$$(10) \quad g_1[X_{-\beta}(Xw), X_{2\alpha+\beta}(X^2v)]g_1^{-1} = \frac{[X_{-\beta}(Xw)X_{-\alpha-\beta}(c_1)X_{-2(\alpha+\beta)}(c_2)X_{-2\alpha-\beta}(X^{-1}c_3), X_{2\alpha+\beta}(X^2v)X_{\beta}(c_4)X_{\alpha+\beta}(Xc_5)X_{2(\alpha+\beta)}(X^2c_6)]}{X_{2\alpha+\beta}(X^2v)X_{\beta}(c_4)X_{\alpha+\beta}(Xc_5)X_{2(\alpha+\beta)}(X^2c_6)},$$

where $c_1 \in V_{-\alpha-\beta} \otimes_A A[X]$, $c_2 \in V_{-2(\alpha+\beta)} \otimes_A A[X]$, etc. One readily sees that conjugating this expression by g_2 does not change its shape, so we can skip this operation. Now, using [LSt, Lemma 2 (2)] again, we write

$$\begin{aligned} X_{2\alpha+\beta}(X^2v) &= [X_{\alpha}(Xv_1), X_{\alpha+\beta}(Xv_2)][X_{2\alpha}(X^2v_3), X_{\beta}(v_4)] \cdot X_{2\alpha+2\beta}(-X^2N_{2\alpha, \beta, 1, 2}(v_3, v_4)) \cdot \\ &\quad \cdot \prod_{i=1}^m ([X_{\alpha}(Xc_i), X_{\beta}(d_i)]X_{\alpha+\beta}(-XN_{\alpha, \beta, 1, 1}(c_i, d_i))X_{2(\alpha+\beta)}(-X^2N_{\alpha, \beta, 2, 2}(c_i, d_i))), \end{aligned}$$

for some $v_1, c_1, \dots, c_m \in V_{\alpha} \otimes_A A[X]$, $v_2 \in V_{\alpha+\beta} \otimes_A A[X]$, $v_3 \in V_{2\alpha} \otimes_A A[X]$, $v_4, d_1, \dots, d_m \in V_{\beta} \otimes_A A[X]$. Now we observe that the commutator of $X_{-2\alpha-\beta}(X^{-1}c_3)$ with any element of $X_{\alpha}(V_{\alpha} \otimes_A XA[X])^{\pm 1}$, $X_{\alpha+\beta}(V_{\alpha+\beta} \otimes_A XA[X])^{\pm 1}$, $X_{\beta}(V_{\beta} \otimes_A A[X])$, $X_{2\alpha}(V_{2\alpha} \otimes_A X^2A[X])$, $X_{2(\alpha+\beta)}(V_{2(\alpha+\beta)} \otimes_A X^2A[X])$ belongs to the product $Y_{\alpha, \beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X])$. Therefore, since $Y_{\alpha, \beta}^+$ is normalized by $X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X])$ by Lemma 4.7 (ii), we can cancel the factor $X_{-2\alpha-\beta}(X^{-1}c_3)$ in (10) with its inverse, so that the result belongs to $Y_{\alpha, \beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X])$. Consequently, we have

$$g[X_{-\beta}(Xw), X_{2\alpha+\beta}(X^2v)]g^{-1} \in Y_{\alpha, \beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X]).$$

Another type of factors occurring in (9) are factors $X_{2\alpha+\beta}(X^2s_i)$, $1 \leq i \leq k$. By the case (i) we have $gX_{2\alpha+\beta}(X^2s_i)g^{-1} \in Y_{\alpha, \beta}^+$.

Finally, consider $g[X_{-\beta}(Xw_i), X_{2\alpha+2\beta}(Xv_i)]g^{-1}$, $1 \leq i \leq k$. By the generalized Chevalley commutator formula, exactly as above we have

$$g[X_{-\beta}(Xw_i), X_{2\alpha+2\beta}(Xv_i)]g^{-1} = \frac{[X_{-\beta}(Xw_i)X_{-\alpha-\beta}(c_{1i})X_{-2(\alpha+\beta)}(c_{2i})X_{-2\alpha-\beta}(X^{-1}c_{3i}), X_{2\alpha+2\beta}(Xv_i)]}{X_{2\alpha+2\beta}(Xv_i)}$$

for some $c_{1i} \in V_{-\alpha-\beta} \otimes_A A[X]$, $c_{2i} \in V_{-2(\alpha+\beta)} \otimes_A A[X]$, etc. Note that

$$[X_{-2\alpha-\beta}(X^{-1}c_{3i}), X_{2\alpha+2\beta}(Xv_i)] = X_{\beta}(N_{-2\alpha-\beta, 2\alpha+2\beta, 1, 1}(c_{3i}, v_i))X_{-2\alpha}(X^{-1}N_{-2\alpha-\beta, 2\alpha+2\beta, 1, 2}(c_{3i}, v_i)).$$

Hence, by Lemma 4.7 we conclude that

$$g[X_{-\beta}(Xw_i), X_{2\alpha+2\beta}(Xv_i)]g^{-1} \in Y_{\alpha, \beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X]).$$

Applying all these results to the expression in (9), we deduce that

$$gX_{\alpha}(V_{\alpha} \otimes_A X^2A[X])g^{-1} \subseteq Y_{\alpha, \beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X]).$$

(iii) To compute $gX_{\alpha}(V_{\alpha} \otimes_A X^2A[X])g^{-1}$, we use the same relations as in (8) to decompose $X_{\alpha}(V_{\alpha} \otimes_A X^2A[X])$:

$$\begin{aligned} X_{\alpha}(V_{\alpha} \otimes_A X^2A[X]) &\subseteq [X_{\alpha+\beta}(V_{\alpha+\beta} \otimes_A XA[X]), X_{-\beta}(V_{-\beta} \otimes_A XA[X])] \cdot \\ &\quad \cdot X_{2\alpha+\beta}(V_{2\alpha+\beta} \otimes_A X^3A[X]) \cdot X_{2\alpha}(V_{2\alpha} \otimes_A X^4A[X]). \end{aligned}$$

By the previous results, we only need to consider g -conjugates of the commutator. We have

$$\begin{aligned} (11) \quad g_1[X_{\alpha+\beta}(V_{\alpha+\beta} \otimes_A XA[X]), X_{-\beta}(V_{-\beta} \otimes_A XA[X])]g_1^{-1} &\subseteq \\ &\subseteq [X_{\alpha+\beta}(V_{\alpha+\beta} \otimes_A XA[X])X_{\beta}(V_{\beta} \otimes_A A[X]), \\ &\quad X_{-\beta}(V_{-\beta} \otimes_A XA[X])X_{-(\alpha+\beta)}(V_{-(\alpha+\beta)} \otimes_A A[X]) \cdot \\ &\quad \cdot X_{-2(\alpha+\beta)}(V_{-2(\alpha+\beta)} \otimes_A A[X])X_{-2\alpha-\beta}(V_{-2\alpha-\beta} \otimes_A X^{-1}A[X])]. \end{aligned}$$

Again, conjugating right-hand side by g_2 does not change the shape of the right-hand side, so it leaves to note that

$$\begin{aligned} & [X_{\alpha+\beta}(V_{\alpha+\beta} \otimes_A XA[X])X_\beta(V_\beta \otimes_A A[X]), X_{-2\alpha-\beta}(V_{-2\alpha-\beta} \otimes_A X^{-1}A[X])] \subseteq \\ & \subseteq \left\langle X_{\alpha+\beta}(V_{\alpha+\beta} \otimes_A XA[X]), X_\beta(V_\beta \otimes_A A[X]), X_{-\alpha}(V_{-\alpha} \otimes_A A[X]), \right. \\ & \left. X_{-2\alpha}(V_{-2\alpha} \otimes_A A[X]), X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X]) \right\rangle \subseteq Y_{\alpha,\beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X]); \end{aligned}$$

then the right-hand side of (11) is contained in $Y_{\alpha,\beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X])$. \square

Lemma 4.9. *Assume that $\Psi = BC_l$, $l \geq 2$. Let $\alpha \in \Psi$ be an extra-short root. Then*

$$\tau_\alpha^{\pm 1}(E_\alpha(A[X], XA[X])) \subseteq (G'_\alpha(A[X]) \cap E(A[X])) \cdot X_{\mp 2\alpha}(X^{-1}V_{\mp 2\alpha}).$$

More precisely, if $\beta \in \Psi$ is another root such that α, β form a system of simple roots for a subsystem of Ψ of type BC_2 , we have

$$\tau_\alpha^{\pm 1}(E_\alpha(A[X], XA[X])) \subseteq (G'_\alpha(A[X]) \cap Y_{\pm\alpha, \pm\beta}^+) \cdot X_{\mp 2\alpha}(X^{-1}V_{\mp 2\alpha}),$$

where the subgroup $Y_{\alpha,\beta}^+$ is defined as in Lemma 4.7.

Proof. The cases of τ_α and $\tau_\alpha^{-1} = \tau_{-\alpha}$ are symmetric, so it is enough to consider τ_α . As in the proof of Lemma 4.6, we conclude that $E_\alpha(A[X], XA[X])$ is generated by the elements of $aX_\alpha(V_\alpha \otimes_A XA[X])X_{2\alpha}(V_{2\alpha} \otimes_A XA[X])a^{-1}$, where $a \in E_\alpha(A)$. By Gauss decomposition in $E_\alpha(A)$, we have $\tau_\alpha(a) = xyzh$, where $x, z \in X_\alpha(V_\alpha \otimes_A XA[X])X_{2\alpha}(V_{2\alpha} \otimes_A X^2A[X])$, $y \in X_{-\alpha}(V_{-\alpha} \otimes_A X^{-1}A[X])X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-2}A[X])$, and $h \in L_\alpha(A)$. Clearly, conjugation by zh preserves the group

$$\tau_\alpha(X_\alpha(V_\alpha \otimes_A XA[X])X_{2\alpha}(V_{2\alpha} \otimes_A XA[X])) = X_\alpha(V_\alpha \otimes_A X^2A[X])X_{2\alpha}(V_{2\alpha} \otimes_A X^3A[X]).$$

By Lemma 4.8 both $yX_\alpha(V_\alpha \otimes_A X^2A[X])y^{-1}$ and $yX_{2\alpha}(V_{2\alpha} \otimes_A X^3A[X])y^{-1}$ are contained in $Y_{\alpha,\beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X])$. Note that by Lemma 4.7 (ii) $X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X])$ normalizes $Y_{\alpha,\beta}^+$. By Lemma 4.7 (i) $x \in Y_{\alpha,\beta}^+$, hence

$$x \cdot Y_{\alpha,\beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X]) \cdot x^{-1} \subseteq Y_{\alpha,\beta}^+ \cdot X_{-2\alpha}(V_{-2\alpha} \otimes_A X^{-1}A[X]).$$

This completes the proof, since $Y_{\alpha,\beta}^+ \subseteq E(A[X])$. \square

4.4. Properties of σ .

Lemma 4.10. *Assume that $m_1(\tilde{\alpha}) = 1$, that is, $\Psi \neq G_2, F_4, E_8, BC_n$. Then $\sigma^{\pm 1}(E(A[X], XA[X])) \subseteq U_1^\mp(A)(L_1(A) \cap E(A))E(A[X], XA[X])$.*

Proof. It is enough to consider the case of σ . For any $\alpha \in \Psi$, the restriction $\sigma|_{E_\alpha(A[X, X^{-1}])}$ coincides with τ_α , $\tau_\alpha^{-1} = \tau_{-\alpha}$, or is trivial. Hence by Lemma 4.6 we have $\sigma^{\pm 1}(E(A[X], XA[X])) \subseteq E(A[X])$. Write $\sigma(g) \in \sigma(E(A[X], XA[X]))$ as $g = g_0g_1$, $g_0 = g(0) \in E(A)$, $g_1 = g(0)^{-1}g \in E(A[X], XA[X])$. Using Gauss decomposition in $G(A)$, write $g_0 = u_1hvu_2$, where $u_1, u_2 \in U_1^-(A)$, $v \in U_1^+(A)$, $h \in L_1(A) \cap E(A)$. Then we have

$$\sigma^{-1}(g) = \sigma^{-1}(g_0)\sigma^{-1}(g_1) \in E(A[X], XA[X])h\sigma^{-1}(v)E(A[X]),$$

where $\sigma^{-1}(v) \in U_1^+(A[X^{-1}], X^{-1}A[X^{-1}])$. On the other hand, by the definition of g , $\sigma^{-1}(g) \in E(A[X], XA[X])$. Consequently,

$$\sigma^{-1}(v) \in U_1^+(A[X^{-1}], X^{-1}A[X^{-1}]) \cap E(A[X]) = \{1\}.$$

Therefore, $v = 1$, and we are done. \square

Lemma 4.11. *Assume that $\Psi = G_2, F_4, E_8$. Then*

$$\sigma^{\pm 1}(E_{\tilde{\alpha}}(A[X], XA[X])) \subseteq E(A[X])X_{\mp \tilde{\alpha}}(X^{-1}V_{\mp \tilde{\alpha}}).$$

Proof. It is enough to consider the case of σ . We have $\sigma|_{E_{\tilde{\alpha}}(E[X, X^{-1}])} = \tau_{\tilde{\alpha}}^2$. By Lemma 4.5 there is a root $\alpha \in \Psi$, such that $\tilde{\alpha}, \alpha$ form a basis of a root subsystem $\Theta = \Theta_{\tilde{\alpha}}$ of type A_2 in Ψ , and the group scheme G contains a simply connected split simple subgroup G_{Θ} , such that $X_{\beta}, \beta\Theta$, are root subgroups of G_{Θ} . Then $\tau_{\tilde{\alpha}}$ is also the restriction to $G'_{\tilde{\alpha}}$ of the automorphism σ_{Θ} of G_{Θ} , defined in the same way as σ with respect to $\alpha_1 = \tilde{\alpha}$, and let L_{Θ} denote the σ_{Θ} -invariant Levi subgroup, the analogue of L_1 in G . Let E_{Θ} denote the elementary subgroup of G_{Θ} . Applying Lemma 4.10 (with both sides inverted) to G_{Θ} instead of G , we deduce that

$$\tau_{\tilde{\alpha}}(E_{\tilde{\alpha}}(A[X], XA[X])) \subseteq E_{\Theta}(A[X], XA[X]) \cdot (L_{\Theta}(A) \cap E_{\Theta}(A)) \cdot U_{\{-\tilde{\alpha}, -\alpha, -\tilde{\alpha}-\alpha\}}(A).$$

Note that, by the very definition, $\tau_{\tilde{\alpha}}(E_{\tilde{\alpha}}(A[X], XA[X])) \subseteq E_{\tilde{\alpha}}(A[X, X^{-1}]) \subseteq G'_{\tilde{\alpha}}(A[X, X^{-1}])$. Since $G(A[X]) \cap G'_{\tilde{\alpha}}(A[X, X^{-1}]) = G'_{\tilde{\alpha}}(A[X])$, we have

$$\tau_{\tilde{\alpha}}(E_{\tilde{\alpha}}(A[X], XA[X])) \subseteq E(A[X], XA[X]) \cdot (L(A) \cap E(A)) \cdot U_{\{-\tilde{\alpha}, -\alpha, -\tilde{\alpha}-\alpha\}}(A) \cap G'_{\tilde{\alpha}}(A[X]).$$

Since $L_{\Theta}U_{\{-\tilde{\alpha}, -\alpha, -\tilde{\alpha}-\alpha\}} \cap G'_{\tilde{\alpha}} = (L_{\Theta} \cap G'_{\tilde{\alpha}})U_{-\tilde{\alpha}}$ by the definitions of $G_{\Theta_{\tilde{\alpha}}}$ and $G'_{\tilde{\alpha}}$ (and, e.g., by [SGA3, Exp. XXVI Prop. 1.20]), we have, in fact,

$$\tau_{\tilde{\alpha}}(E_{\tilde{\alpha}}(A[X], XA[X])) \subseteq E_{\Theta}(A[X], XA[X]) \cdot (L_{\Theta}(A) \cap E_{\Theta}(A)) \cdot X_{-\tilde{\alpha}}(V_{-\tilde{\alpha}}).$$

Then, again by Lemma 4.10,

$$\tau_{\tilde{\alpha}}^2(E_{\tilde{\alpha}}(A[X], XA[X])) \subseteq \sigma_{\Theta}(E_{\Theta}(A[X], XA[X]) \cdot (L_{\Theta}(A) \cap E_{\Theta}(A)) \cdot X_{-\tilde{\alpha}}(V_{-\tilde{\alpha}})) \subseteq E_{\Theta}(A[X])X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}}).$$

Since $E_{\Theta}(A[X]) \subseteq E(A[X])$, we are done. \square

Lemma 4.12. *One has*

$$\sigma^{\pm 1}(E(A[X], XA[X])) \subseteq U_1^{\mp}(A)(L_1(A) \cap E(A))E(A[X], XA[X])X_{\mp\tilde{\alpha}}(X^{-1}V_{\mp\tilde{\alpha}}).$$

Proof. It is enough to consider the case of σ . If $m_1(\tilde{\alpha}) = 1$, the claim follows from Lemma 4.10. Assume $m_1(\tilde{\alpha}) = 2$. By Lemma 3.10, any $x \in E(A[X], XA[X])$ can be presented as a product $x = x_1x_2$, where x_1 is a product of elements of the groups $E_{\beta}(A[X], XA[X])$, where $\beta \in \Psi$ is non-collinear to $\tilde{\alpha}$; and x_2 belongs to $E_{\tilde{\alpha}}(A[X], XA[X])$, or, respectively, $E_{\frac{1}{2}\tilde{\alpha}}(A[X], XA[X])$, if $\frac{1}{2}\tilde{\alpha}$ is a root. Note that $\tilde{\alpha}$ is the only root α such that $m_1(\alpha) = 2$ and σ acts non-trivially on $X_{\pm\alpha}$; and, if $\Psi = BC_l$, the root $\frac{1}{2}\tilde{\alpha} = \alpha$ is the only extra-short root such that σ acts non-trivially on $X_{\pm\alpha}$. Then by Lemmas 4.6, 4.11, and 4.9, we have

$$(12) \quad \sigma^{\pm 1}(E(A[X], XA[X])) \subseteq E(A[X])X_{\mp\tilde{\alpha}}(X^{-1}V_{\mp\tilde{\alpha}}) = E(A)E(A[X], XA[X])X_{\mp\tilde{\alpha}}(X^{-1}V_{\mp\tilde{\alpha}}).$$

Write $g \in \sigma(E(A[X], XA[X]))$ as $g = g_0g_1g_2$ with the components from the respective factors. By Gauss decomposition, we have $g_0 = u_1hvu_2$, where $u_1, u_2 \in U_1^+(A)$, $v \in U_1^-(A)$, $h \in L_1(A) \cap E(A)$. We will compute $\sigma^{-1}(g)$ using this factorization. Inverting both sides of (12), we obtain $\sigma^{-1}(g_1) \in X_{2\tilde{\alpha}}(X^{-1}c)E(A[X])$ for some $c \in V_{2\tilde{\alpha}}$, and hence $\sigma^{-1}(g_1g_2) \in X_{2\tilde{\alpha}}(X^{-1}c)E(A[X])$.

Write

$$u_2 = \prod_{m_1(\alpha)=-1} X_{\alpha}(c_{\alpha}) \cdot X_{-\frac{\tilde{\alpha}}{2}}(d)X_{-\tilde{\alpha}}(e),$$

where c_{α}, d, e belong to the respective root modules, and the factor $X_{-\frac{\tilde{\alpha}}{2}}(d)$ is omitted if $\Psi \neq BC_l$. Then

$$(13) \quad \sigma^{-1}(u_2) = \prod_{\substack{m_1(\alpha)=-1, \\ \alpha \neq -\frac{\tilde{\alpha}}{2}}} X_{\alpha}(Xc_{\alpha}) \cdot X_{-\frac{\tilde{\alpha}}{2}}(Xd)X_{-\tilde{\alpha}}(X^2e).$$

Note that by Lemma 4.7 (with signs changed) if $\Psi = BC_n$, and by Lemma 4.6 if $\Psi = G_2, F_4, E_8$, we have

$$X_{-\frac{\tilde{\alpha}}{2}}(Xd)X_{-\tilde{\alpha}}(X^2e)X_{\tilde{\alpha}}(X^{-1}c) \in X_{\tilde{\alpha}}(X^{-1}c)E(A[X]).$$

By the generalized Chevalley commutator formula, since $\tilde{\alpha} \in \Psi$ is a root of maximal length, we have

$$\prod_{\substack{m_1(\alpha)=-1, \\ \alpha \neq -\frac{\tilde{\alpha}}{2}}} X_\alpha(Xc_\alpha) \cdot X_{\tilde{\alpha}}(X^{-1}c) \in X_{\tilde{\alpha}}(X^{-1}c)E(A[X]).$$

Summing up, we have

$$\sigma^{-1}(u_2g_1g_2) \in X_{\tilde{\alpha}}(X^{-1}c)E(A[X]),$$

and, consequently,

$$\sigma^{-1}(g) \in \sigma^{-1}(u_1h)\sigma^{-1}(v)\sigma^{-1}(u_2g_1g_2) \in E(A[X])\sigma^{-1}(v)X_{\tilde{\alpha}}(X^{-1}c)E(A[X]),$$

where, moreover, $\sigma^{-1}(v) \in U_1^+(A[X^{-1}], X^{-1}A[X^{-1}])$. Recall that, on the other hand, $\sigma^{-1}(g) \in E(A[X], XA[X])$ by the definition of g . Then we have

$$\sigma^{-1}(v)X_{\tilde{\alpha}}(X^{-1}c) \in E(A[X^{-1}], X^{-1}A[X^{-1}]) \cap E(A[X]) = \{1\}.$$

Writing a more detailed factorization for $\sigma^{-1}(v)$ as in (13), we see that this implies $v = 1$. Consequently,

$$g \in U_1^-(A)E(A[X], XA[X])X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}}).$$

□

4.5. Properties of $E^*(A[X, X^{-1}], I \cdot A[X, X^{-1}])$.

Lemma 4.13. *We have*

$$\sigma^{\pm 1}(E(A[X], XA[X]) \cap \ker \rho) \subseteq X_{\mp \tilde{\alpha}}(X^{-1}IV_{\mp \tilde{\alpha}})M_+^*.$$

Consequently, $\sigma(M_+^*M_-^*) = M_+^*M_-^*$.

Proof. To prove the first claim, recall that for any $g \in E(A[X], XA[X])$ we have $\sigma(g) = X_{-\tilde{\alpha}}(X^{-1}u)h$ for some $u \in V_{-\tilde{\alpha}}$, $h \in E(A[X])$, by Lemma 4.12. If, moreover, $\rho(g) = 1$, then $\rho(\sigma(g)) = 1$ as well, hence, $\rho(X_{-\tilde{\alpha}}(X^{-1}u)) = X_{-\tilde{\alpha}}(X^{-1}\rho(u))$ belongs to $E(l[X])$. This implies that $\rho(u) = 0$, or $u \in IV_{-\tilde{\alpha}}$. Automatically, $h \in M_+^*$. The claim is proved.

Now take any $y \in M_+^*$ and $z \in M_-^*$. We can write $y = y_1y_0$, $y_0 = y(0) \in E(A)$, $y_1 = yy_0^{-1} \in E(A[X], XA[X])$. In the same way, $z = z_0z_1$, where $z_0 = z(\infty) \in E(A)$, $z_1 = z(\infty)^{-1}z \in E(A[X^{-1}], X^{-1}A[X^{-1}])$. Clearly, $\rho(y_0) = \rho(y_1) = \rho(z_0) = \rho(z_1) = 1$. By Lemma 3.11 we have $x = y_0z_0 = x^-tx^+ \in U_1^-(I)(L_1(A, I) \cap E(A))U_1^+(I)$. Since $E(A)$ normalizes $E(A[X], XA[X])$ and $E(A[X^{-1}], X^{-1}A[X^{-1}])$, we can rewrite

$$(14) \quad yz = x^-ty_1z_1x^+,$$

for some new $y_1 \in E(A[X], XA[X]) \cap \ker \rho$ and $z_1 \in E(A[X^{-1}], X^{-1}A[X^{-1}]) \cap \ker \rho$.

The first claim of the lemma together with symmetry arguments implies

$$\begin{aligned} \sigma^{\pm 1}(E(A[X^{-1}], X^{-1}A[X^{-1}]) \cap \ker \rho) &\subseteq X_{\pm \tilde{\alpha}}(XIV_{\pm \tilde{\alpha}})M_-^*; \\ \sigma^{\pm 1}(E(A[X], XA[X]) \cap \ker \rho) &\subseteq X_{\mp \tilde{\alpha}}(X^{-1}IV_{\mp \tilde{\alpha}})M_+^*. \end{aligned}$$

Hence we have

$$\sigma^{-1}(y_1z_1) \in M_+^*X_{\tilde{\alpha}}(X^{-1}IV_{\mp \tilde{\alpha}})X_{-\tilde{\alpha}}(XIV_{\pm \tilde{\alpha}})M_-^*.$$

Applying $\tau_{\tilde{\alpha}}$ to the middle factor and using Lemma 3.11 one more time, we deduce that

$$\sigma^{-1}(y_1z_1) \in M_+^*M_-^*.$$

Now one readily sees that $\sigma^{-1}(yz) \in M_+^*M_-^*$. Therefore, $\sigma^{-1}(M_+M_-) \subseteq M_+^*M_-^*$. By symmetry, $\sigma(M_+M_-) \subseteq M_+^*M_-^*$. Hence $\sigma(M_+^*M_-^*) = M_+^*M_-^*$. □

Lemma 4.14. *We have $M_-^*E(A[X]) \subseteq E(A[X])M_-^*$.*

Proof. The group $E(A[X])$ is generated by $U_1^\pm(A[X])$ by the main theorem of [PSt1]. Hence any element of this group is a product of elements of the form $X_\alpha(X^k u)$, for $\alpha \in \Psi$ such that $m_1(\alpha) \neq 0$, and $u \in V_\alpha$, $k \geq 0$. We show by descending induction on k that $X_\alpha(X^k u)zX_\alpha(X^k u)^{-1} \in E(A[X])M_-^*$, for any $z \in M_-^*$. Since M_-^* is normalized by $E(A[X^{-1}])$, the case $k \leq 0$ is clear. Consider the case $k > 0$. We can assume that $z \in E(A[X^{-1}], X^{-1}A[X^{-1}]) \cap \ker \rho$, since $X_\alpha(X^k u)z(0)X_\alpha(X^k u)^{-1} \in E(A[X])$.

By symmetry, we can also assume $\alpha \in \Psi^+$, i.e. $m_1(\alpha) > 0$. We have either $m_1(\alpha) = 1$, or $m_1(\alpha) = 2$, $\alpha = \tilde{\alpha}$. Then we have

$$X_\alpha(X^k u)zX_\alpha(X^k u)^{-1} = \sigma(X_\alpha(X^{k-m_1(\alpha)}u)\sigma^{-1}(z)X_\alpha(X^{k-m_1(\alpha)}u)^{-1}).$$

Since $z \in E(A[X^{-1}], X^{-1}A[X^{-1}]) \cap \ker \rho$, by Lemma 4.13 we have $\sigma^{-1}(z) \in X_{-\tilde{\alpha}}(IXV_{-\tilde{\alpha}})M_-^*$. Then, by the induction hypothesis

$$y = X_\alpha(X^{k-m_1(\alpha)}u)\sigma^{-1}(z)X_\alpha(X^{k-m_1(\alpha)}u)^{-1} \in X_\alpha(X^{k-m_1(\alpha)}u)X_{-\tilde{\alpha}}(IXV_{-\tilde{\alpha}})X_\alpha(X^{k-m_1(\alpha)}u)^{-1}M_-^*.$$

If $\alpha \neq \tilde{\alpha}$ or $k > 1$, then, clearly, $X_\alpha(X^{k-m_1(\alpha)}u)X_{-\tilde{\alpha}}(IXV_{-\tilde{\alpha}})X_\alpha(X^{k-m_1(\alpha)}u)^{-1} \subseteq M_+^*$. Otherwise we have

$$\begin{aligned} X_\alpha(X^{k-m_1(\alpha)}u)X_{-\tilde{\alpha}}(IXV_{-\tilde{\alpha}})X_\alpha(X^{k-m_1(\alpha)}u)^{-1} &= X_{\tilde{\alpha}}(X^{-1}u)X_{-\tilde{\alpha}}(IXV_{-\tilde{\alpha}})X_{\tilde{\alpha}}(X^{-1}u)^{-1} \\ &= \tau_{\tilde{\alpha}}^{-1}(X_{\tilde{\alpha}}(u)X_{-\tilde{\alpha}}(IV_{-\tilde{\alpha}})X_{\tilde{\alpha}}(u)^{-1}) \subseteq \tau_{\tilde{\alpha}}^{-1}(E_{\tilde{\alpha}}(A, I)) \subseteq M_+^*M_-^* \end{aligned}$$

by Lemma 3.11.

Summing up, we have $y \in M_+^*M_-^*$. Hence, by Lemma 4.13 we have

$$X_\alpha(X^k u)zX_\alpha(X^k u)^{-1} = \sigma(y) \in \sigma(M_+^*M_-^*) = M_+^*M_-^*.$$

□

4.6. Decomposition of $E(A[X, X^{-1}])$ and $E^*(A[X, X^{-1}], I \cdot A[X, X^{-1}])$.

Lemma 4.15. *One has*

$$X_{\pm\tilde{\alpha}}(X^{-1}u)E(A[X], XA[X]) \subseteq U_1^\pm(A)(L_1(A) \cap E(A))E(A[X], XA[X])X_{\pm\tilde{\alpha}}(X^{-1}V_{\pm\tilde{\alpha}})X_{\mp\tilde{\alpha}}(XV_{\mp\tilde{\alpha}}),$$

for any $u \in V_{\pm\tilde{\alpha}}$.

Proof. Clearly, it is enough to consider the case of $X_{\tilde{\alpha}}(X^{-1}u)$. Applying Lemma 4.12 two times, we deduce

$$\begin{aligned} X_{\tilde{\alpha}}(X^{-1}u)E(A[X], XA[X]) &= \sigma^{-1}\left(X_{\tilde{\alpha}}(Xu)\sigma(E(A[X], XA[X]))\right) \\ &\subseteq \sigma^{-1}\left(X_{\tilde{\alpha}}(Xu)U_1^-(A)(L_1(A) \cap E(A))E(A[X], XA[X])X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}})\right) \\ &\subseteq \sigma^{-1}\left(U_1^-(A)(L_1(A) \cap E(A))E(A[X], XA[X]) \cdot X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}})\right) \\ &\subseteq E(A[X], XA[X])(L_1(A) \cap E(A))U_1^+(A)E(A[X], XA[X])X_{\tilde{\alpha}}(X^{-1}V_{\tilde{\alpha}})X_{-\tilde{\alpha}}(XV_{-\tilde{\alpha}}) \\ &\subseteq U_1^+(A)(L_1(A) \cap E(A))E(A[X], XA[X])X_{\tilde{\alpha}}(X^{-1}V_{\tilde{\alpha}})X_{-\tilde{\alpha}}(XV_{-\tilde{\alpha}}). \end{aligned}$$

□

Lemma 4.16. *The product*

$$E(A[X])E(A[X^{-1}])E(A[X])$$

is invariant under σ^\pm .

Proof. We consider the case of σ , the case of σ^{-1} is symmetric. Set

$$Z = E(A[X])E(A[X^{-1}])E(A[X]).$$

Since $E(A)$ normalizes $E(A[X], XA[X])$ and $E(A[X^{-1}], X^{-1}A[X^{-1}])$, and we have $E(A[X]) = E(A)E(A[X], XA[X])$, $E(A[X^{-1}]) = E(A)E(A[X^{-1}], X^{-1}A[X^{-1}])$ by Lemma 3.6, we conclude that

$$E(A[X])E(A[X^{-1}])E(A[X]) = E(A[X], XA[X])E(A)E(A[X^{-1}], X^{-1}A[X^{-1}])E(A[X], XA[X]).$$

Since A is semilocal, we have Gauss decomposition [SGA3, Théorème 5.1]

$$E(A) = U_1^+(A)U_1^-(A)EL_1(A)U_1^+(A) = U_1^+(A)EL_1(A)U_1^-(A)U_1^+(A),$$

where we denote $EL_1(A) = L_1(A) \cap E(A)$. Then we have

$$Z = U_1^+(A)EL_1(A)E(A[X], XA[X])E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)E(A[X], XA[X])U_1^+(A).$$

Applying Lemma 4.12, we see that

$$(15) \quad \sigma(Z) \subseteq E(A[X])X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}}) \cdot \sigma\left(E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)\right) \cdot X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}})E(A[X]).$$

Using Lemma 4.15, we compute

$$\begin{aligned} \sigma\left(E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)\right) \cdot X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}}) &= \sigma\left(E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)X_{-\tilde{\alpha}}(XV_{-\tilde{\alpha}})\right) \\ &= \sigma\left(E(A[X^{-1}], X^{-1}A[X^{-1}])X_{-\tilde{\alpha}}(XV_{-\tilde{\alpha}})U_1^-(A)\right) \\ &\subseteq \sigma\left(X_{\tilde{\alpha}}(X^{-1}V_{\tilde{\alpha}})X_{-\tilde{\alpha}}(XV_{-\tilde{\alpha}})E(A[X^{-1}], X^{-1}A[X^{-1}])EL_1(A)U_1^-(A)\right) \\ &\subseteq X_{\tilde{\alpha}}(XV_{\tilde{\alpha}})X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}})\sigma\left(E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)\right)EL_1(A). \end{aligned}$$

Note that we have used the inclusion from Lemma 4.15 with both sides inverted; we can do it since $\tilde{\alpha}$ is a non-multipliable root, and therefore $(X_{\tilde{\alpha}}(XV_{\tilde{\alpha}}))^{-1} = X_{\tilde{\alpha}}(XV_{\tilde{\alpha}})$.

Substituting the previous computation into (15), we obtain

$$(16) \quad \sigma(Z) \subseteq E(A[X]) \cdot X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}})X_{\tilde{\alpha}}(XV_{\tilde{\alpha}})X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}}) \cdot \sigma\left(E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)\right) \cdot E(A[X]).$$

By Gauss decomposition in $E_{\tilde{\alpha}}(A)$, we have

$$\begin{aligned} X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}})X_{\tilde{\alpha}}(XV_{\tilde{\alpha}})X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}}) &= \tau_{\tilde{\alpha}}\left(X_{-\tilde{\alpha}}(V_{-\tilde{\alpha}})X_{\tilde{\alpha}}(V_{\tilde{\alpha}})X_{-\tilde{\alpha}}(V_{-\tilde{\alpha}})\right) \\ &\subseteq \tau_{\tilde{\alpha}}\left((L_{\tilde{\alpha}}(A) \cap E_{\tilde{\alpha}}(A))X_{\tilde{\alpha}}(V_{\tilde{\alpha}})X_{-\tilde{\alpha}}(V_{-\tilde{\alpha}})X_{\tilde{\alpha}}(V_{\tilde{\alpha}})\right) \\ &= (L_{\tilde{\alpha}}(A) \cap E_{\tilde{\alpha}}(A))X_{\tilde{\alpha}}(XV_{\tilde{\alpha}})X_{-\tilde{\alpha}}(X^{-1}V_{-\tilde{\alpha}})X_{\tilde{\alpha}}(XV_{\tilde{\alpha}}). \end{aligned}$$

Substituting this result into (16), we see that to complete the proof of the lemma, it is enough to show that

$$X_{\tilde{\alpha}}(XV_{\tilde{\alpha}})\sigma\left(E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)\right) \subseteq E(A[X^{-1}])E(A[X]).$$

We obtain this inclusion as follows:

$$\begin{aligned} X_{\tilde{\alpha}}(XV_{\tilde{\alpha}})\sigma\left(E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)\right) &= \sigma\left(X_{\tilde{\alpha}}(X^{-1}V_{\tilde{\alpha}})E(A[X^{-1}], X^{-1}A[X^{-1}])U_1^-(A)\right) \\ &= \sigma\left(U_1^-(A)E(A[X^{-1}], X^{-1}A[X^{-1}])\right) \subseteq E(A[X^{-1}])X_{\tilde{\alpha}}(XV_{\tilde{\alpha}}), \end{aligned}$$

again by Lemma 4.12. \square

Lemma 4.17. *We have $E(A[X, X^{-1}]) = E(A[X])E(A[X^{-1}])E(A[X])$.*

Proof. Set $Z = E(A[X])E(A[X^{-1}])E(A[X])$. By [PSt1, Lemma 12] the group $E(A[X, X^{-1}])$ is generated by $U_1^+(A[X, X^{-1}])$ and $U_1^-(A[X, X^{-1}])$. Clearly, it is enough to show that $U_1^{\pm}(A[X^{-1}])Z \subseteq Z$, or even that

$$X_{\alpha}(X^{-k}u)Z \subseteq Z$$

for any $\alpha \in \Psi$ such that $m_1(\alpha) \neq 0$, $u \in V_{\alpha}$, and $k > 0$. We can assume $\alpha \in \Psi^+$ without loss of generality. Then $\sigma^k(X_{\alpha}(X^{-k}u)) = X_{\alpha}(X^{-k+m_1(\alpha)}u) \in E(A[X])$, hence $\sigma^k(X_{\alpha}(X^{-k}u))Z \subseteq Z$. Since Z is σ -invariant by Lemma 4.16, we are done. \square

Lemma 4.18. *One has $M^* = M_+^*M_-^*$.*

Proof. We need to show that $M^* = M_+^*M_-^*$. Let $x \in M^*$. By Lemma 4.17 we have $x = x_1yx_2$, where $x_1, x_2 \in E(A[X])$, $y \in E(A[X^{-1}])$. Since $\rho(x) = 1$, we have $\rho(y) = \rho(x_1)^{-1}\rho(x_2)^{-1} \in E(l[X^{-1}])$. Since $E(l[X^{-1}]) \cap E(l[X]) = E(l)$, we have $\rho(y) \in E(l)$. Then $y \in E(A)M_-^*$. By Lemma 4.14 we have $M_-^*E(A[X]) \subseteq E(A[X])M_-^*$, hence $yx_2 \in E(A[X])M_-^*$, and thus $x = x_1yx_2 \in E(A[X])M_-^*$. Since $\rho(x) = 1$, then $x \in M_+^*M_-^*$. Hence $M^* = M_+^*M_-^*$. \square

Proof of Theorem 4.1. Lemma 4.17 provides the claim of Theorem 4.1 in case where G is a simple reductive group. To prove the general case, we can right away assume that G is semisimple, since the elementary subgroup is contained in the derived subgroup of G . Clearly, we can also assume that G is simply connected. Then the group G is a direct product of Weil restrictions of simple groups defined over finite étale extensions of R by [SGA3, Exp. XIV Prop. 5.10]. This implies the claim. \square

Proof of Corollary 4.2. Lemma 4.18 provides the claim for G simple. In general, as in the proof of Theorem 4.1, we can assume that G is a semisimple reductive group, and the claim holds for the simply connected covering G^{sc} of G . Any $g \in E^*(A[X, X^{-1}], I \cdot A[X, X^{-1}])$ can be lifted to an element $\tilde{g} \in E^{sc}(A[X, X^{-1}])$, where E^{sc} denotes the elementary subgroup of G^{sc} . Then $\rho(\tilde{g}) \in \text{Cent}(G^{sc})(l[X, X^{-1}]) \cap E^{sc}(l[X, X^{-1}])$, where $\rho : A \rightarrow A/I = l$ is the residue homomorphism. Since $\text{Cent}(G^{sc})$ is an étale twisted form of a direct product of groups of the form μ_n , $n \geq 2$, one has $\text{Cent}(G^{sc})(l[X, X^{-1}]) = \text{Cent}(G^{sc})(l)$. Therefore, $\rho(\tilde{g})$ belongs to $E^{sc}(l)$, and lifts to an element of $E^{sc}(A)$. Hence we can assume that

$$\tilde{g} \in E^{sc}(A)E^{sc*}(A[X, X^{-1}], I \cdot A[X, X^{-1}]) = E^{sc}(A)E^{sc*}(A[X], I \cdot A[X])E^{sc*}(A[X^{-1}], I \cdot A[X^{-1}]).$$

This implies that $g \in E^*(A[X], I \cdot A[X]) \cdot E^*(A[X^{-1}], I \cdot A[X^{-1}])$, as required. \square

5. THE MAIN RESULTS

5.1. \mathbb{P}^1 -gluing. The following Theorem was proved in [Su, Th. 5.1] for GL_n , and in [A, Th. 2.16] for most Chevalley groups. It means, essentially, that K_1^G satisfies the gluing property for the standard covering of \mathbb{P}^1 by two copies of \mathbb{A}^1 over an affine base. Indeed, since a reductive group scheme G and its parabolic subgroups are affine schemes, $K_1^G(\mathbb{P}_A^1) \cong K_1^G(A)$ for any commutative ring A .

Theorem 5.1. *Let A be a commutative ring, G a reductive group scheme over A , satisfying the condition (E). Then the sequence of pointed sets*

$$1 \longrightarrow K_1^G(A) \xrightarrow{g \mapsto (g, g)} K_1^G(A[X]) \times K_1^G(A[X^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_1^G(A[X, X^{-1}])$$

is exact. In particular, $G(A[X], XA[X]) \cap E(A[X, X^{-1}]) \subseteq E(A[X])$.

Proof. Clearly, we need to prove the exactness only at the third term. It follows easily from the following statement. For any $x \in G(A[X], XA[X])$, if there exists an element $y \in G(A[X^{-1}])$ such that $xy^{-1} \in E(A[X, X^{-1}])$, then $x \in E(A[X])$. We prove the latter.

By Suslin's local-global principle Lemma 2.1 we can assume that A is local. Let I be the maximal ideal of A , $l = A/I$, $\rho : G(A[X, X^{-1}]) \rightarrow G(l[X, X^{-1}])$ the natural map. By the Margaux–Soulé Theorem 2.3, $G(l[X]) = G(l)E(l[X])$. Since $x \in G(A[X], XA[X])$, we have $\rho(x) \in E(l[X])$, and hence $x \in E(A[X])G(A[X], I \cdot A[X])$. Therefore, we can assume $x \in G(A[X], I \cdot A[X])$ from the start.

Then, by the assumption of the theorem, $\rho(y) \in E(l[X, X^{-1}])$ and hence, using the Margaux–Soulé Theorem 2.3 again,

$$\rho(y) \in G(l[X^{-1}]) \cap E(l[X, X^{-1}]) = G(l)E(l[X^{-1}]) \cap E(l[X, X^{-1}]).$$

Since $G(l) \cap E(l[X, X^{-1}]) = E(l)$ (send X to 1), we have $\rho(y) \in E(l)E(l[X^{-1}]) = E(l[X^{-1}])$, and $y \in E(A[X^{-1}])G(A[X^{-1}], I \cdot A[X^{-1}])$. Adjusting y by the corresponding factor from $E(A[X^{-1}])$, we can assume that $y \in G(A[X^{-1}], I \cdot A[X^{-1}])$ from the start. Then

$$xy^{-1} \in G(A[X, X^{-1}], I \cdot A[X, X^{-1}]) \cap E(A[X, X^{-1}]) = E^*(A[X, X^{-1}], I \cdot A[X, X^{-1}]).$$

Then by Corollary 4.2 we have $xy^{-1} = x_+ x_-$ for some $x_+ \in E(A[X])$, $x_- \in E(A[X^{-1}])$. Therefore, $x_+^{-1}x = x_- y \in G(A[X]) \cap G(A[X^{-1}]) = G(A)$. Hence $x \in G(A)E(A[X])$, and thus $x \in E(A[X])$. \square

The following Lemma is a direct corollary of Theorem 5.1 together with Lemma 3.3; it extends [Su, Corollary 5.7], [A, Prop. 3.3].

Lemma 5.2. *Let A , G be as in Theorem 5.1. Let $x = x(X) \in G(A[X])$ be such that $x(X) \in G(A[X], XA[X])$ and $f \in A[X]$ a monic polynomial. If $F_f(x) \in E(A[X]_f)$, then $x \in E(A[X])$.*

Proof. The proof literally repeats that of [A, Proposition 3.3] (or [Su, Corollary 5.7]), using Lemma 3.3 instead of [A, Lemma 3.2] and Theorem 5.1 instead of [A, Theorem 2.16]. \square

5.2. K_1^G of polynomial rings over a field. The following theorem is an extension of [A, Theorem 3.5] for Chevalley groups. We repeat Abe's proof almost literally, referring to respective facts about isotropic groups (proved in this paper and elsewhere) instead of lemmas on split groups used by Abe.

Theorem 5.3. *Let G be a reductive group scheme over a field k , such that every semisimple normal subgroup of G contains $(\mathbf{G}_m)^2$. Then $G(k[X_1, \dots, X_n]) = G(k)E(k[X_1, \dots, X_n])$ for any $n \geq 1$. In other words, $K_1^G(k) \cong K_1^G(k[X_1, \dots, X_n])$.*

Proof. By Lemma 2.2, it is enough to prove the claim in the case where G is a simply connected semisimple group. The proof goes by induction on n . The case $n = 1$ for G a simple algebraic group (i.e. having an irreducible Dynkin diagram) is the Margaux–Soulé Theorem 2.3.

Assume that the theorem is true for any number of variables less than n , for a fixed field k . Let $x = x(X_1, \dots, X_n) \in G(k[X_1, \dots, X_n])$. We can assume that $x(X_1, \dots, X_{n-1}, 0) = 1$. Next, consider the inclusion $G(k[X_1, \dots, X_n]) \subseteq G(k(X_1, \dots, X_n))$. By [G, Théorème 5.8] and induction on n we have $G(k(X_1, \dots, X_n)) = G(k)E(k(X_1, \dots, X_n))$. We can assume that x lands in $E(k(X_1, \dots, X_n))$ and again $x(X_1, \dots, X_{n-1}, 0) = 1$. Then there exists a polynomial $f \in k[X_1, \dots, X_n]$ such that $x \in E(k[X_1, \dots, X_n]_f)$. Write $f = \sum_{i=0}^m a_i(X_1, \dots, X_{n-1})X_n^i$ so that $g = a_m(X_1, \dots, X_{n-1}) \neq 0$.

Then f can be assumed to be a monic polynomial in X_n over the ring $A = k[X_1, \dots, X_{n-1}]_g$. Then $x \in G(A[X_n], X_n A[X_n]) \cap E(A[X_n]_f)$.

By Lemma 5.2 we have $x \in E(A[X_n])$. If $g \in k$ is a constant, we are done. If g is not a constant, we can assume that g contains the variable X_{n-1} . Applying induction on the number of variables involved in g , we can assume $x(X_1, \dots, X_{n-2}, 0, 0) = 1$. Write $g = \sum_{i=0}^l b_i(X_1, \dots, X_{n-2})X_{n-1}^i$, so that the leading term $h = a_l(X_1, \dots, X_{n-2}) \neq 0$. Then g is a monic polynomial in X_{n-1} over the ring $B = k[X_1, \dots, X_{n-2}, X_n]_h$. Then $x \in E(B[X_{n-1}]_g)$. Applying Lemma 5.2 again, we obtain $x \in E(B[X_{n-1}]) = E(k[X_1, \dots, X_{n-2}, X_{n-1}, X_n]_h)$. By the inductive assumption on the number of variables involved in g , we have then $x \in E(k[X_1, \dots, X_n])$. \square

Corollary 5.4. *Let G be a simply connected semisimple group scheme over a field k , such that every semisimple normal subgroup of G contains $(\mathbf{G}_m)^2$. For any $m, n \geq 0$, there are natural isomorphisms*

$$K_1^G(k) \cong K_1^G(k[Y_1, \dots, Y_m, X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]) \cong K_1^G(k(Y_1, \dots, Y_m, X_1, \dots, X_n)).$$

Proof. We prove the first isomorphism by induction on n , reducing to the case $n = 0$, which is contained in Theorem 5.3. The second isomorphism follows from [G, Théorème 5.8].

Consider the natural map

$$K_1^G(k[Y_1, \dots, Y_m, X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]) \rightarrow K_1^G(k(X_1)[Y_1, \dots, Y_m, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}]).$$

Since, by the inductive hypothesis and [G, Théorème 5.8], we have

$$K_1^G(k) \cong K_1^G(k(X_1)) \cong K_1^G(k(X_1)[Y_1, \dots, Y_m, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}]),$$

it is enough to prove that this map is injective. Set $B = k[Y_1, \dots, Y_m, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}]$. Assume that $g \in G(B[X_1, X_1^{-1}])$ is mapped into $E(k(X_1)[Y_1, \dots, Y_m, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}])$. Then there exists a monic polynomial $f \in k[X_1]$ such that

$$g \in E(k[X_1]_f[X_1^{-1}][Y_1, \dots, Y_m, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}]) = E(B[X_1]_{X_1 f}).$$

Clearly, we can assume that f is not divided by X_1 . Then by Lemma 3.3 there exist $g_1 \in E(B[X_1]_{X_1})$, $g_2 \in E(B[X_1]_f)$ such that $g = g_1 g_2$. The class of g_1 in $K_1^G(B[X_1]_{X_1}) = K_1^G(B[X_1, X_1^{-1}])$ is trivial, hence we can assume $g = g_2$. Since $B[X_1]_{X_1} \cap B[X_1]_f = B$, we have $g \in G(B[X_1])$. By the inductive hypothesis we also have $G(B[X_1]) = G(k)E(B[X_1])$. Hence we can assume that $g \in G(k)$. But then $g \in E(k(Y_1, \dots, Y_m, X_1, \dots, X_n))$ implies $g \in E(k)$. The claim proved. \square

5.3. Homotopy invariance theorem. The following theorem follows from Theorem 5.3 by means of two geometric reduction results (Lindel's lemma and Popescu's theorem).

Theorem 5.5. *Let G be a reductive group scheme over a perfect field k , such that every semisimple normal subgroup of G contains $(\mathbf{G}_m)^2$. Let A be a regular ring containing k . Then for any $n \geq 1$, the inclusion map induces an isomorphism*

$$K_1^G(A) \xrightarrow{\cong} K_1^G(A[X_1, \dots, X_n]).$$

Lemma 5.6. *Let k, A, G be as in Theorem 5.5. Assume in addition that A is of essentially finite type over k . Then for any $n \geq 1$ one has*

$$K_1^G(A) \xrightarrow{\cong} K_1^G(A[X_1, \dots, X_n]).$$

Proof. The proof is essentially the same as the one for GL_n in [V, Theorem 3.1], using the facts about isotropic groups we have proved before. Namely, one proceeds by induction on $\dim A$. By Suslin's local-global principle Lemma 2.1 we can assume A is local. If $\dim A = 0$, we are in the setting of Theorem 5.3. Hence we can assume $\dim A \geq 1$. By Lindel's lemma [L, Lemma and Proposition 2] there exists a subring B of A and an element $h \in B$ such that $B = k[X_1, \dots, X_n]_p$, where p is a prime of $k[X_1, \dots, X_n]$, and $Ah + B = A$, $Ah \cap B = Bh$.

We need to show that $G(A[X_1, \dots, X_n]) = G(A)E(A[X_1, \dots, X_n])$. Take $x(X_1, \dots, X_n) \in G(A[X_1, \dots, X_n])$. We can assume from the start that $x(0, \dots, 0) = 1$. Since $\dim A_h < \dim A$, we have $x(X_1, \dots, X_n) \in G(A_h)E(A_h[X_1, \dots, X_n])$. Since $x(0, \dots, 0) = 1$, we have in fact $x(X_1, \dots, X_n) \in E(A_h[X_1, \dots, X_n])$. Since A is local and regular, we know that h is not a zero divisor in $A[X_1, \dots, X_n]$; hence by Lemma 3.4 (ii) we have

$$x(X_1, \dots, X_n) = y(X_1, \dots, X_n)z(X_1, \dots, X_n)$$

for some $y(X_1, \dots, X_n) \in E(A[X_1, \dots, X_n])$ and $z(X_1, \dots, X_n) \in G(B[X_1, \dots, X_n])$. Clearly, we can assume that $z(0, \dots, 0) = 1$ as well. Since B is a localization of a polynomial ring over k , by Lemma 3.7 and Theorem 5.3 we have $z(X_1, \dots, X_n) \in E(B[X_1, \dots, X_n])$. Therefore, $x(X_1, \dots, X_n) \in E(A[X_1, \dots, X_n])$. \square

Proof of Theorem 5.5. The embedding $k \rightarrow A$ is geometrically regular, since k is perfect [Ma, (28.M), (28.N)]. Then by Popescu's theorem [Po, Sw] A is a filtered direct limit of regular k -algebras essentially of finite type. Since the group scheme G and the unipotent radicals of its parabolic subgroups are finitely presented over k , the functors $G(-)$ and $E(-)$ commute with filtered direct limits. Hence the claim of the theorem follows from Lemma 5.6. \square

5.4. An injectivity property for K_1^G .

Theorem 5.7. *Let G be a reductive group scheme over a infinite perfect field k , such that every semisimple normal subgroup of G contains $(\mathbf{G}_m)^2$. Let A be a local ring of a smooth algebraic variety over k , or, if k is perfect, just a local regular ring containing k . Let K be the field of fractions of A . Then the natural homomorphism*

$$K_1^G(A) \rightarrow K_1^G(K)$$

is injective.

Proof. We can reduce to the case where A is a local ring of a smooth algebraic variety over k in the same way as in the proof of Theorem 5.5, using Popescu's theorem. To show that $K_1^G(A) \rightarrow K_1^G(K)$ is injective for any A of this kind, it is enough to check that the functor K_1^G on the category of commutative k -algebras satisfies the conditions of [CTO, Théorème 1.1]. The condition (P1) of [CTO, Théorème 1.1] is that K_1^G commutes with filtered direct limits; this is clear.

The condition (P2) requires checking that $K_1^G(L[X_1, \dots, X_n]) \rightarrow K_1^G(L(X_1, \dots, X_n))$ has trivial kernel for any field L containing k and any $n \geq 1$. By Theorem 5.3 and induction on n it is enough to check that for any field L as above, if $g \in G(L)$ is mapped into $E(L(X))$, then it belongs to $E(L)$. There is a polynomial $f \in L[X]$ such that $g \in E(L[X]_f)$. Since k is an infinite field, there is $a \in L$ such that $f(a) \neq 0$. The evaluation at $X = a$ then shows that $g \in E(L)$.

The condition (P3) follows from Lemma 3.4 (ii). \square

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